Intrinsic Flat and Gromov-Hausdorff Convergence

Christina Sormani

CUNY GC and Lehman College

Lecture I
Intrinsic Flat and Gromov-Hausdorff Convergence

Lecture 1: Geometric Notions of Convergence
Smooth \( C^k \), Lipschitz (Lip), Gromov-Hausdorff (GH), Sormani-Wenger Intrinsic Flat (SWIF) Convergence

[Gromov Structures-Metriques] [Sormani-Wenger JDG-2011]
Intrinsic Flat and Gromov-Hausdorff Convergence

Lecture 1: Geometric Notions of Convergence
Smooth ($C^k$), Lipschitz (Lip), Gromov-Hausdorff (GH), Sormani-Wenger Intrinsic Flat (SWIF) Convergence
[Gromov Structures-Metriques] [Sormani-Wenger JDG-2011]

Lecture 2: Open Problems about Scalar Curvature
Almost Rigidity of the Positive Mass Theorem
Geometric Stability of the Scalar Torus Rigidity Theorem
Scalar Sphere Rigidity Theorem and more....
[Conjectures on Conv and Scalar (2021) arXiv: 2103.10093]
Intrinsic Flat and Gromov-Hausdorff Convergence

Lecture 1: Geometric Notions of Convergence
Smooth ($C^k$), Lipschitz (Lip), Gromov-Hausdorff (GH), Sormani-Wenger Intrinsic Flat (SWIF) Convergence
[Gromov Structures-Metriques] [Sormani-Wenger JDG-2011]

Lecture 2: Open Problems about Scalar Curvature
Almost Rigidity of the Positive Mass Theorem
Geometric Stability of the Scalar Torus Rigidity Theorem
Scalar Sphere Rigidity Theorem and more....
[Conjectures on Conv and Scalar (2021) arXiv: 2103.10093]

Lectures 3&4: Techniques to Apply to Prove Convergence
Ambrosio-Kirchheim Theory of Integral Currents
Decomposition into Regions with Lakzian Properties with Portegies and Arzela-Ascoli Theorems
Volume Above Distance Below with Allen and Perales
See https://sites.google.com/site/intrinsicflatconvergence/
Lecture 1: Geometric Notions of Convergence

Goal Today: Build Geometric Intuition
Lecture 1: Geometric Notions of Convergence

Goal Today: Build Geometric Intuition
Viewing Riemannian Manifolds as Metric Spaces
Lecture 1: Geometric Notions of Convergence

Goal Today: Build Geometric Intuition
Viewing Riemannian Manifolds as Metric Spaces
and Introducing Integral Current Spaces
Lecture 1: Geometric Notions of Convergence

Goal Today: Build Geometric Intuition
Viewing Riemannian Manifolds as Metric Spaces and Introducing Integral Current Spaces

Notions of Convergence:
Lecture 1: Geometric Notions of Convergence

Goal Today: Build Geometric Intuition
Viewing Riemannian Manifolds as Metric Spaces
and Introducing Integral Current Spaces

Notions of Convergence:
Smooth Convergence ($C^k$),
Lecture 1: Geometric Notions of Convergence

Goal Today: Build Geometric Intuition
Viewing Riemannian Manifolds as Metric Spaces
and Introducing Integral Current Spaces

Notions of Convergence:
Smooth Convergence ($C^k$),
Gromov Lipschitz Convergence (Lip),
Gromov-Hausdorff Convergence (GH),
Sormani-Wenger Intrinsic Flat Convergence (SWIF),
Volume Preserving Intrinsic Flat Convergence (VF),
Allen-Perales-Sormani (VADB) Convergence

Recommended Resources:
[2] Burago-Burago-Ivanov Text
Lecture 1: Geometric Notions of Convergence

Goal Today: Build Geometric Intuition
Viewing Riemannian Manifolds as Metric Spaces and Introducing Integral Current Spaces

Notions of Convergence:
Smooth Convergence ($C^k$),
Gromov Lipschitz Convergence (Lip),
Gromov-Hausdorff Convergence (GH),

Recommended Resources:
[Gromov Structures-Metriques]
[Burago-Burago-Ivanov Text]
[Sormani-Wenger JDG-2011]
[Conjectures on Conv and Scalar (2021) arXiv: 2103.10093]
Lecture 1: Geometric Notions of Convergence

Goal Today: Build Geometric Intuition
Viewing Riemannian Manifolds as Metric Spaces
and Introducing Integral Current Spaces

Notions of Convergence:
Smooth Convergence ($C^k$),
Gromov Lipschitz Convergence (Lip),
Gromov-Hausdorff Convergence (GH),
Sormani-Wenger Intrinsic Flat Convergence (SWIF) or ($\mathcal{F}$),

Recommended Resources:
[2] Burago-Burago-Ivanov Text
Lecture 1: Geometric Notions of Convergence

Goal Today: Build Geometric Intuition
Viewing Riemannian Manifolds as Metric Spaces
and Introducing Integral Current Spaces

Notions of Convergence:
Smooth Convergence ($C^k$),
Gromov Lipschitz Convergence (Lip),
Gromov-Hausdorff Convergence (GH),
Sormani-Wenger Intrinsic Flat Convergence (SWIF) or ($\mathcal{F}$),
Volume Preserving Intrinsic Flat Convergence ($\mathcal{VF}$)

Recommended Resources:
[Gromov Structures-Metriques]
[Burago-Burago-Ivanov Text]
[Sormani-Wenger JDG-2011]
[Conjectures on Conv and Scalar (2021) arXiv: 2103.10093]
Lecture 1: Geometric Notions of Convergence

Goal Today: Build Geometric Intuition
Viewing Riemannian Manifolds as Metric Spaces and Introducing Integral Current Spaces

Notions of Convergence:
Smooth Convergence ($C^k$),
Gromov Lipschitz Convergence (Lip),
Gromov-Hausdorff Convergence (GH),
Sormani-Wenger Intrinsic Flat Convergence (SWIF) or ($\mathcal{F}$),
Volume Preserving Intrinsic Flat Convergence ($\mathcal{VF}$)
Allen-Perales-Sormani (VADB) Convergence

Recommended Resources:
[Gromov Structures-Metriques]
[Burago-Burago-Ivanov Text]
[Sormani-Wenger JDG-2011]
[Conjectures on Conv and Scalar (2021) arXiv: 2103.10093]
Lecture 1: Geometric Notions of Convergence

Goal Today: Build Geometric Intuition
Viewing Riemannian Manifolds as Metric Spaces
and Introducing Integral Current Spaces

Notions of Convergence:
Smooth Convergence ($C^k$),
Gromov Lipschitz Convergence (Lip),
Gromov-Hausdorff Convergence (GH),
Sormani-Wenger Intrinsic Flat Convergence (SWIF) or ($\mathcal{F}$),
Volume Preserving Intrinsic Flat Convergence ($\mathcal{VF}$)
Allen-Perales-Sormani (VADB) Convergence

Recommended Resources:
[Gromov Structures-Metriques]
[Burago-Burago-Ivanov Text]
[Sormani-Wenger JDG-2011]
[Conjectures on Conv and Scalar (2021) arXiv: 2103.10093]
A Riemannian Manifold is a Smooth Metric Space

A Riemannian manifold \((M^m, g)\) is a metric space \((M, dg)\)
A Riemannian Manifold is a Smooth Metric Space

A Riemannian manifold \((M^m, g)\) is a metric space \((M, d_g)\) with a smooth collection of charts allowing us
A Riemannian Manifold is a Smooth Metric Space

A Riemannian manifold \((M^m, g)\) is a metric space \((M, d_g)\) with a smooth collection of charts allowing us to define tangent vectors in tangent planes at each point and
A Riemannian Manifold is a Smooth Metric Space

A Riemannian manifold \((M^m, g)\) is a metric space \((M, d_g)\) with a smooth collection of charts allowing us to define tangent vectors in tangent planes at each point and a metric tensor \(g\) which is an inner product on tangent vectors s.t.

\[
d_g(p, q) = \inf \{ L_g(C) : C(0) = p, \ C(1) = q \}
\]

where \(L_g(C) = \int_0^1 \sqrt{g(C'(s), C'(s))} \, ds\)
A Riemannian Manifold is a Smooth Metric Space

A Riemannian manifold \((M^m, g)\) is a metric space \((M, d_g)\) with a smooth collection of charts allowing us to define tangent vectors in tangent planes at each point and a metric tensor \(g\) which is an inner product on tangent vectors s.t.

\[
d_g(p, q) = \inf \{ L_g(C) : C(0) = p, \ C(1) = q \}
\]

where \(L_g(C) = \int_0^1 g(C'(s), C'(s))^{1/2} ds\)

When \(M\) is compact, \(\exists\) a geodesic, \(\gamma_{p,q}\) s.t. \(L(\gamma_{p,q}) = d(p, q)\).

A sphere with a bump: A torus:
$C^k$ Limits of Sequences of Riemannian Manifolds
$C^k$ Limits of Sequences of Riemannian Manifolds

A $C^k$ smooth limit $(M_\infty, g_\infty)$ is diffeomorphic to the sequence with diffeomorphisms $\psi_j : M_\infty \to M_j$ s.t. $\psi_j^* g_j \to g_\infty$ $C^k$ smoothly on $M_\infty$.

\[
d_\infty(x, y) = \lim_{j \to \infty} d_j(\psi_j(x), \psi_j(y)).
\]

\[
\text{and } \text{Vol}(B_p(R)) = \lim_{j \to \infty} \text{Vol}(B_{\psi_j(p)}(R)).
\]

In fact:

\[
d_{\text{Lip}}((M_j, d_j), (M_\infty, d_\infty)) = \log(\max\{\text{dil}(\psi_j), \text{dil}(\psi_j^{-1})\}) \to 0
\]

where $\text{dil}(\psi_j) = \sup\{d_j(\psi_j(x), \psi_j(y)) : x \neq y\}$.
$C^k$ Limits of Sequences of Riemannian Manifolds

A $C^k$ smooth limit $(M_\infty, g_\infty)$ is diffeomorphic to the sequence with diffeomorphisms $\psi_j : M_\infty \rightarrow M_j$ s.t. $\psi_j^* g_j \rightarrow g_\infty$ $C^k$ smoothly on $M_\infty$.

Thus: $d_\infty(x, y) = \lim_{j \rightarrow \infty} d_j(\psi_j(x), \psi_j(y))$.

and $\text{Vol}(B_p(R)) = \lim_{j \rightarrow \infty} \text{Vol}(B_{\psi_j(p)}(R))$.
$C^k$ Limits of Sequences of Riemannian Manifolds

A smooth limit $(M_\infty, g_\infty)$ is diffeomorphic to the sequence with diffeomorphisms $\psi_j : M_\infty \to M_j$ s.t. $\psi_j^* g_j \to g_\infty$ smoothly on $M_\infty$.

Thus:

$$d_\infty(x, y) = \lim_{j \to \infty} d_j(\psi_j(x), \psi_j(y)).$$

and $\text{Vol}(B_p(R)) = \lim_{j \to \infty} \text{Vol}(B_{\psi_j(p)}(R))$

In fact $C^0$ Convergence $\implies$ Gromov Lipschitz (Lip) Convergence:

$$d_{\text{Lip}}((M_j, d_j), (M_\infty, d_\infty)) = \log(\max\{\text{dil}(\psi_j), \text{dil}(\psi_j^{-1})\}) \to 0$$

where $\text{dil}(\psi_j) = \sup\{d_j(\psi_j(x), \psi_j(y))/d_\infty(x, y) : x \neq y\}$

which is well defined for biLipschitz sequences of metric spaces.
A Gromov-Lipschitz limit \((M_\infty, d_\infty)\) of \((M_j, d_j)\) has biLipschitz maps \(\psi_j : M_\infty \to M_j\) s.t. \(dil(\psi_j) \to 1\) and \(dil(\psi_j^{-1}) \to 1\).

Thus: \(d_\infty(x, y) = \lim_{j \to \infty} d_j(\psi_j(x), \psi_j(y))\).

and \(\text{Vol}(B_p(R)) = \lim_{j \to \infty} \text{Vol}(B_{\psi_j(p)}(R))\).
A Gromov-Lipschitz limit \((M_\infty, d_\infty)\) of \((M_j, d_j)\) has biLipschitz maps \(\psi_j : M_\infty \to M_j\) s.t. \(dil(\psi_j) \to 1\) and \(dil(\psi_j^{-1}) \to 1\).

Thus: \[d_\infty(x, y) = \lim_{j \to \infty} d_j(\psi_j(x), \psi_j(y)).\]

And \[\text{Vol}(B_p(R)) = \lim_{j \to \infty} \text{Vol}(B_{\psi_j(p)}(R))\]

Examples with no \(C^k\) or Lip limit:
Lip Limits of Sequences of Metric Spaces

A Gromov-Lipschitz limit \((M_\infty, d_\infty)\) of \((M_j, d_j)\) has biLipschitz maps \(\psi_j : M_\infty \rightarrow M_j\) s.t. \(dil(\psi_j) \rightarrow 1\) and \(dil(\psi_j^{-1}) \rightarrow 1\).

Thus: \(d_\infty(x, y) = \lim_{j \rightarrow \infty} d_j(\psi_j(x), \psi_j(y))\).

and \(\text{Vol}(B_p(R)) = \lim_{j \rightarrow \infty} \text{Vol}(B_{\psi_j(p)}(R))\)

Examples with no \(C^k\) or Lip limit: \(\lim_{j \rightarrow \infty} \text{Vol}(B_{\psi_j(p)}(R)) = 0\)
Gromov Hausdorff Limits via Almost Isometries

**Gromov’s Defn:** Compact metric spaces \((X_j, d_j) \xrightarrow{\text{GH}} (X_\infty, d_\infty)\)

iff \(\exists \varepsilon_j\)-almost isometries \(\psi_j : X_\infty \to X_j\) where \(\varepsilon_j \to 0\).

This means that \(\psi_j\) are \(\varepsilon_j\)-almost distance preserving:

\[
|d_\infty(x, y) - d_j(\psi_j(x), \psi_j(y))| < \varepsilon_j \quad \forall x, y \in X_\infty
\]

and \(\varepsilon_j\)-almost onto: \(X_j \subset T_{\varepsilon_j}(\psi_j(X_\infty))\).
Gromov Hausdorff Limits via Almost Isometries

**Gromov’s Defn:** Compact metric spaces \((X_j, d_j) \xrightarrow{GH} (X_\infty, d_\infty)\)

iff \(\exists \varepsilon_j\)-almost isometries \(\psi_j : X_\infty \to X_j\) where \(\varepsilon_j \to 0\).

This means that \(\psi_j\) are \(\varepsilon_j\)-almost distance preserving:

\[
|d_\infty(x, y) - d_j(\psi_j(x), \psi_j(y))| < \varepsilon_j \quad \forall x, y \in X_\infty
\]

and \(\varepsilon_j\)-almost onto: \(X_j \subset T_{\varepsilon_j}(\psi_j(X_\infty))\).

Here our rainbow drawn \(\psi_j\) are not onto nor continuous:
Gromov Hausdorff Limits via Nets

Gromov’s Prop: Compact \((X_j, d_j) \xrightarrow{\text{GH}} (X_\infty, d_\infty)\) iff

\(\exists \varepsilon_j \to 0\) and there exist \(\varepsilon_j\)-nets \(S_j \subset X_j \subset T_{\varepsilon_j}(S_j)\) which are finite and \(\varepsilon_j\)-almost distance preserving bijections, \(\psi_j : S_\infty \to S_j\), s.t.

\[|d_\infty(x, y) - d_j(\psi_j(x), \psi_j(y))| < \varepsilon_j \quad \forall x, y \in S_\infty.\]
Gromov Hausdorff Limits via Nets

**Gromov’s Prop:** Compact $(X_j, d_j) \xrightarrow{GH} (X_\infty, d_\infty)$ iff

$\exists \varepsilon_j \to 0$ and there exist $\varepsilon_j$-nets $S_j \subset X_j \subset T_{\varepsilon_j}(S_j)$ which are finite and $\varepsilon_j$-almost distance preserving bijections, $\psi_j : S_\infty \to S_j$, s.t.

$$|d_\infty(x, y) - d_j(\psi_j(x), \psi_j(y))| < \varepsilon_j \quad \forall x, y \in S_\infty.$$
Gromov Hausdorff Limits via Nets

**Gromov's Prop:** Compact \((X_j, d_j) \xrightarrow{GH} (X_\infty, d_\infty)\) iff

\[ \exists \varepsilon_j \to 0 \text{ and there exist } \varepsilon_j\text{-nets } S_j \subset X_j \subset T_{\varepsilon_j}(S_j) \text{ which are finite and } \varepsilon_j\text{-almost distance preserving bijections, } \psi_j : S_\infty \to S_j, \text{ s.t.} \]

\[ |d_\infty(x, y) - d_j(\psi_j(x), \psi_j(y))| < \varepsilon_j \quad \forall x, y \in S_\infty. \]
Gromov’s Compactness Theorem

If \((X_j, d_j)\) are compact metric spaces with \(\text{Diam}(X_j) \leq D\) such that \(\forall r > 0 \exists \text{ } r\text{-nets } S^r_j\) with cardinality \(N(r)\) not depending on \(j\) then a subsequence \((X_{j_k}, d_{j_k}) \overset{\text{GH}}{\longrightarrow} (X_\infty, d_\infty)\) where \(X_\infty\) is compact.

\[
\begin{array}{cccc}
\text{Initial shapes} & \quad & \text{Intermediate shapes} & \quad & \text{Final shape}
\end{array}
\]
Gromov’s Compactness Theorem

If \((X_j, d_j)\) are compact metric spaces with \(\text{Diam}(X_j) \leq D\) such that \(\forall r > 0 \exists r\)-nets \(S_j^r\) with cardinality \(N(r)\) not depending on \(j\) then a subsequence \((X_{j_k}, d_{j_k}) \xrightarrow{\text{GH}} (X_\infty, d_\infty)\) where \(X_\infty\) is compact.

Pf: Fix \(r > 0\): \(d_j\) restricted to \(S_j^r \times S_j^r\) is an \(N(r) \times N(r)\) matrix.
Gromov’s Compactness Theorem

If \((X_j, d_j)\) are compact metric spaces with \(\text{Diam}(X_j) \leq D\) such that
\[
\forall r > 0 \ \exists \ r\text{-nets } S'_j \text{ with cardinality } N(r) \text{ not depending on } j
\]
then a subsequence \((X_{j_k}, d_{j_k}) \xrightarrow{\text{GH}} (X_\infty, d_\infty)\) where \(X_\infty\) is compact.

Pf: Fix \(r > 0\): \(d_j\) restricted to \(S'_j \times S'_j\) is an \(N(r) \times N(r)\) matrix. Taking a seq \(r \to 0\) diagonalize the conv subsequences of matrices.
Gromov’s Compactness Theorem

If \((X_j, d_j)\) are compact metric spaces with \(\text{Diam}(X_j) \leq D\) such that \(\forall r > 0 \exists r\)-nets \(S_j^r\) with cardinality \(N(r)\) not depending on \(j\) then a subsequence \((X_{j_k}, d_{j_k}) \xrightarrow{\text{GH}} (X_\infty, d_\infty)\) where \(X_\infty\) is compact.

Pf: Fix \(r > 0\): \(d_j\) restricted to \(S_j^r \times S_j^r\) is an \(N(r) \times N(r)\) matrix. Taking a seq \(r \to 0\) diagonalize the conv subsequences of matrices. This gives a countable pseudometric space with \(r\)-nets of card \(N(r)\).
Gromov’s Compactness Theorem

If \((X_j, d_j)\) are compact metric spaces with \(\text{Diam}(X_j) \leq D\) such that
\[
\forall r > 0 \exists \text{ } r\text{-nets } S^r_j \text{ with cardinality } N(r) \text{ not depending on } j
\]
then a subsequence \((X_{j_k}, d_{j_k}) \xrightarrow{GH} (X_\infty, d_\infty)\) where \(X_\infty\) is compact.

Pf: Fix \(r > 0\): \(d_j\) restricted to \(S^r_j \times S^r_j\) is an \(N(r) \times N(r)\) matrix.
Taking a seq \(r \to 0\) diagonalize the conv subsequences of matrices.
This gives a countable pseudometric space with \(r\text{-nets}\) of card \(N(r)\)
\(X_\infty\) is the metric completion of this pseudometric space. \(\square\)
Gromov’s Compactness Theorem

If \((X_j, d_j)\) are compact metric spaces with \(\text{Diam}(X_j) \leq D\) such that 
\[
\forall r > 0 \exists \text{ } r\text{-nets } S_j^r \text{ with cardinality } N(r) \text{ not depending on } j
\]
then a subsequence \((X_{j_k}, d_{j_k}) \xrightarrow{\text{GH}} (X_\infty, d_\infty)\) where \(X_\infty\) is compact.

\[
\begin{array}{ccc}
\text{\includegraphics[width=0.3\textwidth]{image1}} & \quad & \text{\includegraphics[width=0.3\textwidth]{image2}} \\
\text{\includegraphics[width=0.3\textwidth]{image3}} & \quad & \text{\includegraphics[width=0.3\textwidth]{image4}} \\
\end{array}
\]

Pf: Fix \(r > 0\): \(d_j\) restricted to \(S_j^r \times S_j^r\) is an \(N(r) \times N(r)\) matrix. 
Taking a seq \(r \to 0\) diagonalize the conv subsequences of matrices. 
This gives a countable pseudometric space with \(r\text{-nets}\) of card \(N(r)\) 
\(X_\infty\) is the metric completion of this pseudometric space. \(\square\)

Gromov Thm: \(\text{Diam}(M_j) \leq D\) and \(\text{Ricci}_j \geq H \implies \exists M_{j_k} \xrightarrow{\text{GH}} X_\infty\)
GH as Intrinsic Hausdorff Distance

The Gromov-Hausdorff distance between compact spaces $M_i$ is:

$$d_{GH}(M_1, M_2) = \inf \left\{ d^Z_H( \varphi_1(M_1), \varphi_2(M_2)) \mid \varphi_i : M_i \to Z \right\}$$

where the infimum is taken over all compact metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i^m \to Z$. Here: $d^Z_H( \varphi_1(M_1), \varphi_2(M_2))$ is the Hausdorff distance:

$$= \inf \left\{ R : \varphi_1(M_1) \subset T_R(\varphi_2(M_2)), \varphi_2(M_2) \subset T_R(\varphi_1(M_1)) \right\}$$
**GH as Intrinsic Hausdorff Distance**

The *Gromov-Hausdorff distance* between compact spaces $M_i$ is:

$$d_{GH}(M_1, M_2) = \inf \left\{ d^Z_H(\varphi_1(M_1), \varphi_2(M_2)) \mid \varphi_i : M_i \to Z \right\}$$

where the infimum is taken over all compact metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i \to Z$.

Here: $d^Z_H(\varphi_1(M_1), \varphi_2(M_2))$ is the Hausdorff distance:

$$= \inf \left\{ R : \varphi_1(M_1) \subset T_R(\varphi_2(M_2)), \varphi_2(M_2) \subset T_R(\varphi_1(M_1)) \right\}$$

**Gromov:** $d_{GH}(M_2, M_1) < \epsilon \iff \exists 2\epsilon$-almost isom $\psi : M_2 \to M_1$. 
The Gromov-Hausdorff distance between compact spaces $M_i$ is:

$$d_{GH}(M_1, M_2) = \inf \left\{ d^Z_H(\varphi_1(M_1), \varphi_2(M_2)) \mid \varphi_i : M_i \to Z \right\}$$

where the infimum is taken over all compact metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i^m \to Z$.

Here: $d^Z_H(\varphi_1(M_1), \varphi_2(M_2))$ is the Hausdorff distance:

$$= \inf \left\{ R : \varphi_1(M_1) \subset T_R(\varphi_2(M_2)), \varphi_2(M_2) \subset T_R(\varphi_1(M_1)) \right\}$$

Gromov: $d_{GH}(M_2, M_1) < \epsilon \implies \exists 2\epsilon$-almost isom $\psi : M_2 \to M_1$.

Hint: Let $\psi(p)$ be a point $q \in M_1$ s.t. $d_Z(\varphi_2(p), \varphi_1(q))$ is min.
**GH as Intrinsic Hausdorff Distance**

The *Gromov-Hausdorff distance* between compact spaces $M_i$ is:

$$d_{GH}(M_1, M_2) = \inf \left\{ d_Z^Z(\varphi_1(M_1), \varphi_2(M_2)) \mid \varphi_i : M_i \to Z \right\}$$

where the infimum is taken over all compact metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i \to Z$.

Here: $d_Z^Z(\varphi_1(M_1), \varphi_2(M_2))$ is the Hausdorff distance:

$$= \inf \{ R : \varphi_1(M_1) \subset T_R(\varphi_2(M_2)), \varphi_2(M_2) \subset T_R(\varphi_1(M_1)) \}$$

**Gromov:** $d_{GH}(M_2, M_1) < \epsilon \implies \exists 2\epsilon$-almost isom $\psi : M_2 \to M_1$.

**Hint:** Let $\psi(p)$ be a point $q \in M_1$ s.t. $d_Z(\varphi_2(p), \varphi_1(q))$ is min.

**Gromov:** $d_{GH}(M_2, M_1) = 0 \implies \exists$ an isometry $\psi : M_2 \to M_1$. 
The Gromov-Hausdorff distance between compact spaces $M_i$ is:

$$d_{GH}(M_1, M_2) = \inf \left\{ d_{\mathcal{H}}(\varphi_1(M_1), \varphi_2(M_2)) \mid \varphi_i : M_i \to Z \right\}$$

where the infimum is taken over all compact metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M^m_i \to Z$.

**Gromov:** $d_{GH}(M_2, M_1) < \epsilon \implies \exists 2\epsilon$-almost isom $\psi : M_2 \to M_1$.

**Gromov:** $d_{GH}(M_2, M_1) = 0 \implies \exists$ an isometry $\psi : M_2 \to M_1$. 
GH as Intrinsic Hausdorff Distance

The Gromov-Hausdorff distance between compact spaces $M_i$ is:

$$d_{GH}(M_1, M_2) = \inf \left\{ d_{Z}^H(\varphi_1(M_1), \varphi_2(M_2)) \mid \varphi_i : M_i \to Z \right\}$$

where the infimum is taken over all compact metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i \to Z$.

**Gromov:** $d_{GH}(M_2, M_1) < \epsilon \iff \exists 2\epsilon$-almost isom $\psi : M_2 \to M_1$.

**Gromov:** $d_{GH}(M_2, M_1) = 0 \iff \exists$ an isometry $\psi : M_2 \to M_1$.

**Gromov:** $\exists \epsilon$-alm isom $\psi_j : M_\infty \to M_j \implies d_{GH}(M_j, M_\infty) < 2\epsilon$.

Must construct a metric space $Z$. Can use nets with bridges.
GH as Intrinsic Hausdorff Distance

The Gromov-Hausdorff distance between compact spaces $M_i$ is:

$$d_{GH}(M_1, M_2) = \inf \left\{ d^Z_H(\varphi_1(M_1), \varphi_2(M_2)) \mid \varphi_i : M_i \to Z \right\}$$

where the infimum is taken over all compact metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i \to Z$.

Gromov: $d_{GH}(M_2, M_1) < \epsilon \implies \exists 2\epsilon$-almost isom $\psi : M_2 \to M_1$.

Gromov: $d_{GH}(M_2, M_1) = 0 \implies \exists$ an isometry $\psi : M_2 \to M_1$.

Gromov: $\exists \epsilon$-alm isom $\psi_j : M_{\infty} \to M_j \implies d_{GH}(M_j, M_{\infty}) < 2\epsilon$.

Must construct a metric space $Z$. Can use nets with bridges.

Gromov: If $M_j \xrightarrow{GH} M_{\infty}$ compact then $\exists$ compact $Z$ and $\exists$ dist. pres. maps $\varphi_j : M_j \to Z$ s.t. $d^Z_H(\varphi_j(M_j), \varphi_{\infty}(M_{\infty})) \to 0$. 
GH as Intrinsic Hausdorff Distance

The Gromov-Hausdorff distance between compact spaces $M_i$ is:

$$d_{GH}(M_1, M_2) = \inf \left\{ d^Z_H(\varphi_1(M_1), \varphi_2(M_2)) \mid \varphi_i : M_i \to Z \right\}$$

where the infimum is taken over all compact metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i^m \to Z$.

Gromov: $d_{GH}(M_2, M_1) < \epsilon \implies \exists 2\epsilon$-almost isom $\psi : M_2 \to M_1$.

Gromov: $d_{GH}(M_2, M_1) = 0 \implies \exists$ an isometry $\psi : M_2 \to M_1$.

Gromov: $\exists \epsilon$-alm isom $\psi_j : M_\infty \to M_j \implies d_{GH}(M_j, M_\infty) < 2\epsilon$.

Must construct a metric space $Z$. Can use nets with bridges.

Gromov: If $M_j \xrightarrow{GH} M_\infty$ compact then $\exists$ compact $Z$ and $\exists$ dist. pres. maps $\varphi_j : M_j \to Z$ s.t. $d^Z_H(\varphi_j(M_j), \varphi_\infty(M_\infty)) \to 0$.

Thus, there is a uniform $N(r) = \text{number of pts in an } r\text{-net of } M_j$. 
More Sequences of Riemannian Manifolds

The Lip and GH limit is a manifold with a conical singularity. The GH limit is a smooth manifold with a point singularity with infinite topological type. See Sormani-Wei-JDG.

Gromov: if \( \exists \text{GH lim} \) then \( N(r) \) is uniform. No GH lim here!!!

Ilmanen: \( \exists \) spheres \( M^3 \) with Scalar \( > 0 \) and inc many wells!!!

So GH only works well for \( \text{Ric} \geq H \). We need a new convergence!!!
More Sequences of Riemannian Manifolds

The Lip and GH limit is a manifold with a conical singularity.
More Sequences of Riemannian Manifolds

The Lip and GH limit is a manifold with a conical singularity.

Gromov: if $\exists$ GH lim then $N(r)$ is uniform.

Ilmanen: $\exists$ spheres $M_3$ with $\text{Scalar} > 0$ and inc many wells!!!

So GH only works well for $\text{Ric} \geq H$. We need a new convergence!!!
More Sequences of Riemannian Manifolds

The Lip and GH limit is a manifold with a conical singularity.

The GH limit is a smooth manifold with a point singularity with infinite topological type. See Sormani-Wei-JDG.
More Sequences of Riemannian Manifolds

The Lip and GH limit is a manifold with a conical singularity.

The GH limit is a smooth manifold with a point singularity with infinite topological type. See Sormani-Wei-JDG.
More Sequences of Riemannian Manifolds

The Lip and GH limit is a manifold with a conical singularity.

The GH limit is a smooth manifold with a point singularity with infinite topological type. See Sormani-Wei-JDG.

**Gromov**: if $\exists$ GH lim then $N(r)$ is uniform.
More Sequences of Riemannian Manifolds

The Lip and GH limit is a manifold with a conical singularity.

The GH limit is a smooth manifold with a point singularity with infinite topological type. See Sormani-Wei-JDG.

\[ \text{Gromov: if } \exists \text{ GH lim then } N(r) \text{ is uniform. No GH lim here!!!} \]
More Sequences of Riemannian Manifolds

The Lip and GH limit is a manifold with a conical singularity.

The GH limit is a smooth manifold with a point singularity with infinite topological type. See Sormani-Wei-JDG.

Gromov: if $\exists$ GH lim then $N(r)$ is uniform. No GH lim here!!!
Ilmanen: $\exists$ spheres $M^3$ with $Scalar > 0$ and inc many wells!!!
So GH only works well for $Ric \geq H$. We need a new convergence!!!
GH as Intrinsic Hausdorff Distance

Recall: The Gromov-Hausdorff distance between $M_i^m$ is:

$$d_{GH}(M_1^m, M_2^m) = \inf \left\{ d_Z^Z(\varphi_1(M_1^m), \varphi_2(M_2^m)) \mid \varphi_i : M_i^m \to Z \right\}$$

where the infimum is taken over all compact metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i^m \to Z$:

Here: $d_Z^Z(\varphi_1(M_1^m), \varphi_2(M_2^m))$ is the Hausdorff distance:

$$= \inf \left\{ R : \varphi_1(M_1^m) \subset T_R(\varphi_2(M_2^m)), \varphi_2(M_2^m) \subset T_R(\varphi_1(M_1^m)) \right\}$$
GH as Intrinsic Hausdorff Distance

Recall: The Gromov-Hausdorff distance between \( M_i^m \) is:

\[
d_{GH}(M_1^m, M_2^m) = \inf \left\{ d_Z^H(\varphi_1(M_1^m), \varphi_2(M_2^m)) \mid \varphi_i : M_i^m \rightarrow Z \right\}
\]

where the infimum is taken over all compact metric spaces, \( Z \), and over all distance preserving maps \( \varphi_i : M_i^m \rightarrow Z \):

Here: \( d_Z^H(\varphi_1(M_1^m), \varphi_2(M_2^m)) \) is the Hausdorff distance:

\[
= \inf \left\{ R : \varphi_1(M_1^m) \subset T_R(\varphi_2(M_2^m)), \varphi_2(M_2^m) \subset T_R(\varphi_1(M_1^m)) \right\}
\]

where \( R \) is as large as the depth of a well:
Sormani-Wenger: Intrinsic Flat Distance

The *intrinsic flat distance* between oriented manifolds $M_i^m$ is:

$$d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d_F^Z(\varphi_1\#[[M_1^m]], \varphi_2\#[[M_2^m]]) \mid \varphi_i : M_i^m \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i^m \to Z$.

Here: $d_F^Z(\varphi_1\#[[M_1^m]], \varphi_2\#[[M_2^m]])$ is the Federer-Fleming Flat dist

$$= \inf \left\{ \text{length} (A) + \text{area} (B) : A + \partial B = \varphi_1\#[[M_1^m]] - \varphi_2\#[[M_2^m]] \right\}$$
Sormani-Wenger: Intrinsic Flat Distance

The *intrinsic flat distance* between oriented manifolds $M_i^m$ is:

$$d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d_Z^F (\varphi_1\#[[M_1^m]], \varphi_2\#[[M_2^m]]) \mid \varphi_i : M_i^m \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i^m \to Z$.

Here: $d_Z^F (\varphi_1\#[[M_1^m]], \varphi_2\#[[M_2^m]])$ is the Federer-Fleming Flat dist

$$= \inf \left\{ \text{length} (A) + \text{area} (B) : A + \partial B = \varphi_1\#[[M_1^m]] - \varphi_2\#[[M_2^m]] \right\}$$

This is defined rigorously using [Ambrosio-Kirchheim] for $M_j = (X_j, d_j, T_j)$ which are metric spaces with biLipschitz charts and an integral current structure $T_j$ such that $\text{set}(T_j) = X_j$.

Ambriosio-Kirchheim Theory will be covered in Lecture 3.
The *intrinsic flat distance* between oriented manifolds $M_i^m$ is:

$$d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d_Z^F (\varphi_1#[[M_1^m]], \varphi_2#[[M_2^m]]) \mid \varphi_i : M_i^m \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i^m \to Z$. 

![Diagram of oriented manifolds $M_1$, $M_2$, and $Z$ with distance preserving maps $\varphi_1$ and $\varphi_2$.]
Sormani-Wenger: Intrinsic Flat Distance

The intrinsic flat distance between oriented manifolds $M_i^m$ is:

$$d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d^Z_F (\varphi_1\#[[M_1^m]], \varphi_2\#[[M_2^m]]) \mid \varphi_i : M_i^m \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i^m \to Z$.

Here: $d^Z_F (\varphi_1\#[[M_1^m]], \varphi_2\#[[M_2^m]])$ is the Federer-Fleming Flat dist

$$= \inf \left\{ M_{area}(A) + M_{vol}(B) : A + \partial B = \varphi_1\#[[M_1^m]] - \varphi_2\#[[M_2^m]] \right\}$$
The *intrinsic flat distance* between oriented manifolds $M_i^m$ is:

$$d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d_Z^F (\varphi_1 \#[[M_1^m]], \varphi_2 \#[[M_2^m]]) \mid \varphi_i : M_i^m \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i^m \to Z$.

Here: $d_Z^F (\varphi_1 \#[[M_1^m]], \varphi_2 \#[[M_2^m]])$ is the Federer-Fleming Flat dist

$$= \inf \left\{ \underbrace{M_{\text{area}}(A)}_{\varphi_1 \#[[M_1^m]]} + \underbrace{M_{\text{vol}}(B)}_{\varphi_2 \#[[M_2^m]]} : A + \partial B = \varphi_1 \#[[M_1^m]] - \varphi_2 \#[[M_2^m]] \right\}$$
Defn [SW]: \( M_j \xrightarrow{\text{SWIF}} M_{SWIF} \) iff \( d_{SWIF}(M_j, M_{SWIF}) \to 0 \).

Observe how regions of small volume disappear.
Defn [SW]: \( M_j \xrightarrow{\text{SWIF}} M_{SWIF} \) iff \( d_{SWIF}(M_j, M_{SWIF}) \to 0 \).

Observe how regions of small volume disappear. So the limits may not be connected metric spaces:
Sormani-Wenger: Intrinsic Flat Limits

Defn [SW]: $M_j \xrightarrow{\text{SWIF}} M_{SWIF}$ iff $d_{SWIF}(M_j, M_{SWIF}) \to 0$.

Observe how regions of small volume disappear. So the limits may not be connected metric spaces:

The limit spaces are called integral current spaces $(X, d, T)$: they have countably many biLip charts and a notion of integration over those charts called the integral current structure, $T$. 
Sormani-Wenger: Integral Current Spaces

A sequence of compact Riemannian manifolds can converge in the intrinsic flat (SWIF) sense to the following limit which is an integral current space:

![Integral Current Space Diagram](image)
A sequence of compact Riemannian manifolds can converge in the intrinsic flat (SWIF) sense to the following limit which is an integral current space:

**Defn:** An Integral Current Space \((X,d,T)\) is \(m\)-rectifiable (which means it has countably many bi-Lipschitz charts of the same dimension \(m\) as the original sequence)
A sequence of compact Riemannian manifolds can converge in the intrinsic flat (SWIF) sense to the following limit which is an integral current space:

**Defn:** An Integral Current Space \((X,d,T)\) is \(m\)-rectifiable (which means it has countably many bi-Lipschitz charts of the same dimension \(m\) as the original sequence) and it has a well defined \((m-1)\)-rectifiable boundary.
A sequence of compact Riemannian manifolds can converge in the intrinsic flat (SWIF) sense to the following limit which is an integral current space:

**Defn:** An Integral Current Space \((X,d,T)\) is \(m\)-rectifiable (which means it has countably many bi-Lipschitz charts of the same dimension \(m\) as the original sequence) and it has a well defined \((m-1)\)-rectifiable boundary. The charts are oriented and have integer valued weights, \(\theta\).
A sequence of compact Riemannian manifolds can converge in the intrinsic flat (SWIF) sense to the following limit which is an integral current space:

**Defn:** An Integral Current Space \((X, d, T)\) is m-rectifiable (which means it has countably many bi-Lipschitz charts of the same dimension \(m\) as the original sequence) and it has a well defined \((m-1)\)-rectifiable boundary. The charts are oriented and have integer valued weights, \(\theta\), and are used to define the integral current structure, \(T\),
A sequence of compact Riemannian manifolds can converge in the intrinsic flat (SWIF) sense to the following limit which is an integral current space:

**Defn:** An Integral Current Space \((X,d,T)\) is \(m\)-rectifiable (which means it has countably many bi-Lipschitz charts of the same dimension \(m\) as the original sequence) and it has a well defined \((m-1)\)-rectifiable boundary. The charts are oriented and have integer valued weights, \(\theta\), and are used to define the integral current structure, \(T\), and a measure \(\|T\| = \theta \lambda \mathcal{H}^m\) where \(\lambda\) is the area factor. In Lesson 3 we will define \(T\) using Ambrosio-Kirchheim Theory.
Sormani-Wenger: Intrinsic Flat Distance

The **intrinsic flat distance** between integral current spaces $M_i^m = (X_i, d_i, T_i)$ which we will define carefully in Lecture 3 is:

$$d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d^Z_F (\varphi_1#T_1, \varphi_2#T_2) \mid \varphi_i : M_i^m \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i^m \to Z$.

Here: $d^Z_F (\varphi_1#T_1, \varphi_2#T_2)$ is the Federer-Fleming Flat dist

$$= \inf \left\{ \text{M} \left( \begin{array}{c} \text{area} \end{array} \right) + \text{M} \left( \begin{array}{c} \text{vol} \end{array} \right) : \text{A} + \partial \text{B} = \varphi_1#T_1 - \varphi_2#T_2 \right\}$$
More Intrinsic Flat Limits

Defn [SW]: $M_j \xrightarrow{\text{SWIF}} M_{\text{SWIF}}$ iff $d_{\text{SWIF}}(M_j, M_{\text{SWIF}}) \to 0$
where $M_\infty$ is an integral current space.

What about collapsing tori??
What about spheres shrinking to a point??
If $\text{Vol}(M_j) \to 0$ does the sequence disappear??

We say $M_m \xrightarrow{\text{SWIF}} 0_m$ where $0_m$ is the zero integral current space.
More Intrinsic Flat Limits

**Defn [SW]:** \( M_j \xrightarrow{\text{SWIF}} M_{\text{SWIF}} \) iff \( d_{\text{SWIF}}(M_j, M_{\text{SWIF}}) \to 0 \) where \( M_\infty \) is an integral current space.

What about collapsing tori??
More Intrinsic Flat Limits

**Defn [SW]:** \( M_j \xrightarrow{\text{SWIF}} M_{\text{SWIF}} \) iff \( d_{\text{SWIF}}(M_j, M_{\text{SWIF}}) \to 0 \)

where \( M_\infty \) is an integral current space.

What about collapsing tori??
What about spheres shrinking to a point??
More Intrinsic Flat Limits

**Defn [SW]:** $M_j \xrightarrow{\text{SWIF}} M_{\text{SWIF}}$ iff $d_{\text{SWIF}}(M_j, M_{\text{SWIF}}) \to 0$

where $M_{\infty}$ is an integral current space.

What about collapsing tori??
What about spheres shrinking to a point?
If $Vol(M_j) \to 0$ does the sequence disappear???
More Intrinsic Flat Limits

**Defn [SW]:** \( M_j \xrightarrow{\text{SWIF}} M_{\text{SWIF}} \) iff \( d_{\text{SWIF}}(M_j, M_{\text{SWIF}}) \to 0 \)
where \( M_\infty \) is an integral current space.

What about collapsing tori??
What about spheres shrinking to a point?
If \( \text{Vol}(M_j) \to 0 \) does the sequence disappear???

We say \( M_j \xrightarrow{m \text{ SWIF}} 0^m \) where \( 0^m \) is the zero integral current space.
SWIF convergence to the $0^m$ space

**Defn [SW]:** We say $M_j \xrightarrow{\text{SWIF}} 0$ iff $d_{\text{SWIF}}(M_j^m, 0^m) \to 0$ where

$$d_{\text{SWIF}}(M_j^m, 0^m) = \inf \left\{ d_Z^F(\varphi_1#, [[M_j^m]], [[0]]) \mid \varphi_j : M_j^m \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_j : M_j^m \to Z$.

Here: $d_Z^F(\varphi_1#, [[M_j^m]], [[0]])$ is Federer-Fleming Flat dist:

$$= \inf \left\{ M(A) + M(B) : A + \partial B = \varphi_j#([[M_j^m]] - [[0]]) \right\}$$
SWIF convergence to the $0^m$ space

**Defn [SW]:** We say $M_j \xrightarrow{\text{SWIF}} 0$ iff $d_{\text{SWIF}}(M_j^m, 0^m) \to 0$ where

$$d_{\text{SWIF}}(M_j^m, 0^m) = \inf \left\{ d^Z_F\left(\varphi_1\#[[M_j^m]], [[0]]\right) \mid \varphi_j : M_j^m \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_j : M_j^m \to Z$.

Here: $d^Z_F\left(\varphi_1\#[[M_j^m]], [[0]]\right)$ is Federer-Fleming Flat dist:

$$= \inf \left\{ M([A]) + M([B]) : A + \partial B = \varphi_1\#[[M_j^m]] - [[0]] \right\}$$

**Thm [SW]:** If $\text{Vol}(M_j) \to 0$ then $M_j^m \xrightarrow{\text{SWIF}} 0^m$. 
SWIF convergence to the $0^m$ space

**Defn [SW]:** We say $M_j \xrightarrow{SWIF} 0$ iff $d_{SWIF}(M_j^m, 0^m) \rightarrow 0$ where

$$d_{SWIF}(M_j^m, 0^m) = \inf \left\{ d_F(\varphi_1\#[[M_j^m]], [[0]]) : \varphi_j : M_j^m \rightarrow Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_j : M_j^m \rightarrow Z$.

Here: $d_F(\varphi_1\#[[M_j^m]], [[0]])$ is Federer-Fleming Flat dist:

$$= \inf \left\{ M(A) + M(B) : A + \partial B = \varphi_j\#[[M_j^m]] - [[0]] \right\}$$

**Thm [SW]:** If Vol($M_j$) $\rightarrow 0$ then $M_j^m \xrightarrow{SWIF} 0^m$.

**Pf:** Take $Z = M_j$, $\varphi_j = id$, $A = \varphi_j\#[[M_j^m]]$, and $B = 0$. □
SWIF convergence to the $0^m$ space

**Defn [SW]:** We say $M_j \xrightarrow{\text{SWIF}} 0$ iff $d_{\text{SWIF}}(M_j^m, 0^m) \to 0$ where

$$d_{\text{SWIF}}(M_j^m, 0^m) = \inf \left\{ d_F (\varphi_1 \#[[M_j^m]], [[0]]) \mid \varphi_j : M_j^m \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_j : M_j^m \to Z$.

Here: $d_F (\varphi_1 \#[[M_j^m]], [[0]])$ is Federer-Fleming Flat dist:

$$= \inf \left\{ M(A) + M(B) : A + \partial B = \varphi_1 \#[[M_j^m]] - [[0]] \right\}$$

**Thm [SW]:** If $\text{Vol}(M_j) \to 0$ then $M_j^m \xrightarrow{\text{SWIF}} 0^m$.

**Pf:** Take $Z = M_j$, $\varphi_j = id$, $A = \varphi_j \#[[M_j^m]]$, and $B = 0$. □

In Lecture 3: this notation will be explained and this definition will be made rigorous using Ambrosio-Kirchheim Theory.
SWIF Compactness Theorems

Thm [SW]: If $M_j \xrightarrow{GH} M_{GH}$ and $\text{Vol}(M_j) \leq V_0$ and $\text{Vol}(\partial M_j) \leq A_0$ then $\exists M_{j_k} \xrightarrow{\text{SWIF}} M_{SWIF}$ where $M_{SWIF} \subset M_{GH}$ or $M_{SWIF} = 0$. 

Pf: Gromov $\phi_j: M_j \to \mathbb{Z}$ and Ambrosio-Kirchheim Compactness.

Thm [SW]: If $\text{Vol}(M_j) \geq V_1$ and $\text{Ricci}_j \geq 0$ then $M_{SWIF} = M_{GH}$.

Ilmanen: $\exists M_{3j}$ with $\text{Scal}_j \geq 0$ and inc many wells with no GH lim.

Wenger Compactness Thm: If $\text{Diam}(M_j) \leq D$ and $\text{Vol}(M_j) \leq V$ then $\exists M_{j_k} \xrightarrow{\text{SWIF}} M_{SWIF}$ where possibly $M_{SWIF} = 0$. 

SWIF Compactness Theorems

**Thm [SW]**: If $M_j \xrightarrow{GH} M_{GH}$ and $\text{Vol}(M_j) \leq V_0$ and $\text{Vol}(\partial M_j) \leq A_0$ then $\exists M_{j_k} \xrightarrow{SWIF} M_{SWIF}$ where $M_{SWIF} \subset M_{GH}$ or $M_{SWIF} = 0$.

Pf: Gromov $\varphi_j : M_j \to Z$ and Ambrosio-Kirchheim Compactness.
SWIF Compactness Theorems

**Thm [SW]:** If $M_j \xrightarrow{\text{GH}} M_{GH}$ and $\text{Vol}(M_j) \leq V_0$ and $\text{Vol}(\partial M_j) \leq A_0$ then $\exists M_{j_k} \xrightarrow{\text{SWIF}} M_{SWIF}$ where $M_{SWIF} \subset M_{GH}$ or $M_{SWIF} = 0$.

**Pf:** Gromov $\varphi_j : M_j \rightarrow Z$ and Ambrosio-Kirchheim Compactness.
SWIF Compactness Theorems

**Thm [SW]:** If $M_j \xrightarrow{GH} M_{GH}$ and $\text{Vol}(M_j) \leq V_0$ and $\text{Vol}(\partial M_j) \leq A_0$ then \( \exists M_{j_k} \xrightarrow{SWIF} M_{SWIF} \) where $M_{SWIF} \subset M_{GH}$ or $M_{SWIF} = 0$.

Pf: Gromov $\varphi_j : M_j \rightarrow Z$ and Ambrosio-Kirchheim Compactness.

**Thm [SW]:** If $\text{Vol}(M_j) \geq V_1$ and $\text{Ricci}_{ij} \geq 0$ then $M_{SWIF} = M_{GH}$.
SWIF Compactness Theorems

Thm [SW]: If $M_j \xrightarrow{GH} M_{GH}$ and $\text{Vol}(M_j) \leq V_0$ and $\text{Vol}(\partial M_j) \leq A_0$ then $\exists M_{j_k} \xrightarrow{SWIF} M_{SWIF}$ where $M_{SWIF} \subset M_{GH}$ or $M_{SWIF} = 0$.

Pf: Gromov $\varphi_j : M_j \rightarrow Z$ and Ambrosio-Kirchheim Compactness.

Thm [SW]: If $\text{Vol}(M_j) \geq V_1$ and $\text{Ricci}_{ij} \geq 0$ then $M_{SWIF} = M_{GH}$.

Ilmanen: $\exists M_j^3$ with $\text{Scal}_j \geq 0$ and inc many wells with no GH lim.
SWIF Compactness Theorems

Thm [SW]: If \( M_j \xrightarrow{\text{GH}} M_{GH} \) and \( \text{Vol}(M_j) \leq V_0 \) and \( \text{Vol}(\partial M_j) \leq A_0 \) then \( \exists M_{j_k} \xrightarrow{\text{SWIF}} M_{SWIF} \) where \( M_{SWIF} \subset M_{GH} \) or \( M_{SWIF} = 0 \).

Pf: Gromov \( \varphi_j : M_j \rightarrow Z \) and Ambrosio-Kirchheim Compactness.

Thm [SW]: If \( \text{Vol}(M_j) \geq V_1 \) and \( \text{Ricci}_j \geq 0 \) then \( M_{SWIF} = M_{GH} \).

Ilmanen: \( \exists M_j^3 \) with \( \text{Scal}_j \geq 0 \) and inc many wells with no GH lim.

Wenger Compactness Thm: If \( \text{Diam}(M_j) \leq D \) and \( \text{Vol}(M_j) \leq V \) then \( \exists M_{j_k} \xrightarrow{\text{SWIF}} M_{SWIF} \) where possibly \( M_{SWIF} = 0 \).
SWIF Compactness Theorems

**Thm [SW]:** If $M_j \xrightarrow{\text{GH}} M_{GH}$ and $\text{Vol}(M_j) \leq V_0$ and $\text{Vol}(\partial M_j) \leq A_0$ then $\exists M_{j_k} \xrightarrow{\text{SWIF}} M_{SWIF}$ where $M_{SWIF} \subset M_{GH}$ or $M_{SWIF} = 0$.

**Pf:** Gromov $\varphi_j : M_j \to Z$ and Ambrosio-Kirchheim Compactness.

**Thm [SW]:** If $\text{Vol}(M_j) \geq V_1$ and $\text{Ricci}_j \geq 0$ then $M_{SWIF} = M_{GH}$.

**Ilmanen:** $\exists M_j^3$ with $\text{Scal}_j \geq 0$ and inc many wells with no GH lim.

**Wenger Compactness Thm:** If $\text{Diam}(M_j) \leq D$ and $\text{Vol}(M_j) \leq V$ then $\exists M_{j_k} \xrightarrow{\text{SWIF}} M_{SWIF}$ where possibly $M_{SWIF} = 0$. 
IAS Emerging Topic Conjecture [Gromov-S]

Suppose $M_j^3$ have $\text{Vol}(M_j^3) \leq V$ and $\text{Diam}(M_j^3) \leq D$
IAS Emerging Topic Conjecture [Gromov-S]

Suppose $M_j^3$ have $\text{Vol}(M_j^3) \leq V$ and $\text{Diam}(M_j^3) \leq D$

by [Wenger]: subseq $M_j \xrightarrow{\text{SWIF}} M_\infty$ possibly 0.

Conjecture:
If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$ where $\text{MinA}_j = \min \{\text{Area}(\Sigma) : \text{closed min surfaces } \Sigma \subset M_j^3\}$
Then $M_\infty$ has generalized "Scalar $\geq 0$"

Furthermore:
we believe that we have $M_j \xrightarrow{\text{VF}} M_\infty$:
Volume Preserving Intrinsic Flat Convergence:
$M_j \xrightarrow{\text{SWIF}} M_\infty$ and $\lim_{j \to \infty} \text{Vol}(M_j) = M_{\text{SWIF}}$.

In Lecture 2 we will discuss this conjecture in depth and related open problems at various levels and present examples and partial solutions.

In Lectures 3&4 we will rigorously define integral current spaces, their masses, SWIF and VF convergence, and techniques for proving the open problems.
IAS Emerging Topic Conjecture [Gromov-S]

**Suppose** $M_j^3$ have $\text{Vol}(M_j^3) \leq V$ and $\text{Diam}(M_j^3) \leq D$

by [Wenger]: subseq $M_j \xrightarrow{\text{SWIF}} M_\infty$ possibly 0.
and by [SW]: $\lim \inf_{j \to \infty} \text{Vol}(M_j) \geq M(M_\infty)$.

Conjecture: If in addition we have $\text{Scalar}(M_j^3) \geq 0$ and $\text{MinA}(M_j^3) \geq A$
where $\text{MinA}(M_j^3) = \min$\{Area($\Sigma$): closed min surfaces $\Sigma \subset M_j^3$\}

Then $M_\infty$ has generalized "Scalar $\geq 0".$

Furthermore: we believe that we have $M_j \xrightarrow{\text{VF}} M_\infty$:
Volume Preserving Intrinsic Flat Convergence:
$M_j \xrightarrow{\text{SWIF}} M_\infty$ and $\lim_{j \to \infty} \text{Vol}(M_j) = M(M_\infty)$.

In Lecture 2 we will discuss this conjecture in depth and related open problems at various levels and present examples and partial solutions.
In Lectures 3&4 we will rigorously define integral current spaces, their masses, SWIF and VF convergence, and techniques for proving the open problems.
IAS Emerging Topic Conjecture [Gromov-S]

Suppose $M_j^3$ have $\text{Vol}(M_j^3) \leq V$ and $\text{Diam}(M_j^3) \leq D$

by [Wenger]: subseq $M_j \xrightarrow{\text{SWIF}} M_\infty$ possibly 0.
and by [SW]: $\lim \inf_{j \to \infty} \text{Vol}(M_j) \geq \mathbf{M}(M_\infty)$.
and by [SW]: If $M_\infty \neq 0$ then it is $m$-rectifiable.
IAS Emerging Topic Conjecture [Gromov-S]

Suppose $M_j^3$ have $\text{Vol}(M_j^3) \leq V$ and $\text{Diam}(M_j^3) \leq D$

by [Wenger]: subseq $M_j \xrightarrow{\text{SWIF}} M_\infty$ possibly 0.

and by [SW]: $\liminf_{j \to \infty} \text{Vol}(M_j) \geq M(M_\infty)$.

and by [SW]: If $M_\infty \neq 0$ then it is m-rectifiable.

Conjecture: If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$
IAS Emerging Topic Conjecture [Gromov-S]

Suppose $M^3_j$ have $\text{Vol}(M^3_j) \leq V$ and $\text{Diam}(M^3_j) \leq D$

by [Wenger]: subseq $M_j \overset{\text{SWIF}}{\longrightarrow} M_\infty$ possibly 0.
and by [SW]: $\lim inf_{j\to\infty} \text{Vol}(M_j) \geq \mathcal{M}(M_\infty)$.
and by [SW]: If $M_\infty \neq 0$ then it is m-rectifiable.

Conjecture: If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$
where $\text{MinA}_j = \min\{\text{Area}(\Sigma) : \text{closed min surfaces } \Sigma \subset M^3_j\}$
IAS Emerging Topic Conjecture [Gromov-S]

**Suppose** $M_j^3$ have $\text{Vol}(M_j^3) \leq V$ and $\text{Diam}(M_j^3) \leq D$

by [Wenger]: subseq $M_j \xrightarrow{\text{SWIF}} M_\infty$ possibly 0.

and by [SW]: $\lim \inf_{j \to \infty} \text{Vol}(M_j) \geq M(M_\infty)$.

and by [SW]: If $M_\infty \neq 0$ then it is m-rectifiable.

**Conjecture:** If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$

where $\text{MinA}_j = \min\{\text{Area}(\Sigma) : \text{closed min surfaces } \Sigma \subset M_j^3\}$

Then $M_\infty$ has generalized “$\text{Scalar} \geq 0$”
IAS Emerging Topic Conjecture [Gromov-S]

Suppose $M_j^3$ have $\text{Vol}(M_j^3) \leq V$ and $\text{Diam}(M_j^3) \leq D$

by [Wenger]: subseq $M_j \xrightarrow{\text{SWIF}} M_\infty$ possibly 0.

and by [SW]: $\liminf_{j \to \infty} \text{Vol}(M_j) \geq M(M_\infty)$.

and by [SW]: If $M_\infty \neq 0$ then it is m-rectifiable.

**Conjecture:** If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$

where $\text{MinA}_j = \min\{\text{Area}(\Sigma) : \text{closed min surfaces } \Sigma \subset M_j^3\}$

Then $M_\infty$ has generalized “Scalar $\geq 0$”

Furthermore: we believe that we have $M_j \xrightarrow{\nu,F} M_\infty$:

*Volume Preserving Intrinsic Flat Convergence:*

$M_j \xrightarrow{\text{SWIF}} M_\infty$ and $\lim_{j \to \infty} \text{Vol}(M_j) = M(M_{\text{SWIF}})$.

In Lecture 2 we will discuss this conjecture in depth and related open problems at various levels and present examples and partial solutions.

In Lectures 3&4 we will rigorously define integral current spaces, their masses, SWIF and VF convergence, and techniques for proving the open problems.
IAS Emerging Topic Conjecture [Gromov-S]

**Suppose** $M_j^3$ have $\text{Vol}(M_j^3) \leq V$ and $\text{Diam}(M_j^3) \leq D$

by [Wenger]: subseq $M_j \xrightarrow{\text{SWIF}} M_\infty$ possibly 0.

and by [SW]: $\liminf_{j \to \infty} \text{Vol}(M_j) \geq M(M_\infty)$.

and by [SW]: If $M_\infty \neq 0$ then it is $m$-rectifiable.

**Conjecture:** If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$

where $\text{MinA}_j = \min\{\text{Area}(\Sigma) : \text{closed min surfaces } \Sigma \subset M_j^3\}$

Then $M_\infty$ has generalized “$\text{Scalar} \geq 0$”

**Furthermore:** we believe that we have $M_j \xrightarrow{\nu, \mathcal{F}} M_\infty$:

*Volume Preserving Intrinsic Flat Convergence:*

$M_j \xrightarrow{\text{SWIF}} M_\infty$ and $\lim_{j \to \infty} \text{Vol}(M_j) = M(M_{SWIF})$.

**In Lecture 2** we will discuss this conjecture in depth and related open problems at various levels and present examples and partial solutions.
IAS Emerging Topic Conjecture [Gromov-S]

Suppose $M_j^3$ have $\text{Vol}(M_j^3) \leq V$ and $\text{Diam}(M_j^3) \leq D$ by [Wenger]: subseq $M_j \xrightarrow{\text{SWIF}} M_\infty$ possibly 0.

and by [SW]: $\lim \inf_{j \to \infty} \text{Vol}(M_j) \geq M(M_\infty)$.

and by [SW]: If $M_\infty \neq 0$ then it is $m$-rectifiable.

Conjecture: If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$ where $\text{MinA}_j = \min\{\text{Area}(\Sigma) : \text{closed min surfaces } \Sigma \subset M_j^3\}$

Then $M_\infty$ has generalized “$\text{Scalar} \geq 0$”

Furthermore: we believe that we have $M_j \xrightarrow{\mathcal{VF}} M_\infty$:

Volume Preserving Intrinsic Flat Convergence:

$M_j \xrightarrow{\text{SWIF}} M_\infty$ and $\lim_{j \to \infty} \text{Vol}(M_j) = M(M_{\text{SWIF}})$.

In Lecture 2 we will discuss this conjecture in depth and related open problems at various levels and present examples and partial solutions.

In Lectures 3&4 we will rigorously define integral current spaces, their masses, SWIF and $\mathcal{VF}$ convergence, and techniques for proving the open problems.
Defns of $\mathcal{VF}$ and VADB Convergence

**Defn:** *Volume Preserving Intrinsic Flat Conv:* $M_j \xrightarrow{\mathcal{VF}} M_\infty$

if $M_j \xrightarrow{\text{SWIF}} M_\infty$ and $\lim_{j \to \infty} \text{Vol}(M_j) = \mathcal{M}(M_{SWIF})$. 

In Lecture 4: we will learn that $\mathcal{VF}$ convergence implies measured convergence and other strong consequences of this notion that can be proven using Ambrosio-Kirchheim Theory.

The following theorem enables us to prove $\mathcal{VF}$ convergence for sequences of compact oriented Riemannian manifolds:


$M_j \xrightarrow{\text{VADB}} M_\infty \Rightarrow M_j \xrightarrow{\mathcal{VF}} M_\infty$.

**Defn:** *Volume Above Distance Below Conv:* $M_j \xrightarrow{\text{VADB}} M_\infty$ if $\text{Vol}(M_j) \to \text{Vol}(M_\infty)$ and $\exists D > 0$ s.t. $\text{Diam}(M_j) \leq D$ and $\exists C_1 \psi_j: M_\infty \to M_j$ s.t. $d_j(\psi_j(p), \psi_j(q)) \geq d_\infty(p, q) \forall p, q \in M_\infty$.

Open Question: If $\text{Scal}_j \geq 0$ and $M_j \xrightarrow{\text{VADB}} M_\infty$ then which properties of nonnegative scalar curvature hold on $M_\infty$?
Defns of $\mathcal{VF}$ and VADB Convergence

Defn: *Volume Preserving Intrinsic Flat Conv:* $M_j \xrightarrow{\mathcal{VF}} M_\infty$
if $M_j \xrightarrow{\text{SWIF}} M_\infty$ and $\lim_{j \to \infty} \text{Vol}(M_j) = \mathcal{M}(M_{\text{SWIF}})$.

In Lecture 4: we will learn that $\mathcal{VF}$ convergence implies measured convergence and other strong consequences of this notion that can be proven using Ambrosio-Kirchheim Theory.
Defns of $\mathcal{VF}$ and VADB Convergence

**Defn:** Volume Preserving Intrinsic Flat Conv: \( M_j \xrightarrow{\mathcal{VF}} M_\infty \)

if \( M_j \xrightarrow{\text{SWIF}} M_\infty \) and \( \lim_{j \to \infty} \text{Vol}(M_j) = \text{M}(M_{\text{SWIF}}) \).

In Lecture 4: we will learn that $\mathcal{VF}$ convergence implies measured convergence and other strong consequences of this notion that can be proven using Ambrosio-Kirchheim Theory.

*The following theorem enables us to prove $\mathcal{VF}$ convergence for sequences of compact oriented Riemannian manifolds:*
Defns of $\mathcal{VF}$ and VADB Convergence

**Defn:** Volume Preserving Intrinsic Flat Conv: $M_j \xrightarrow{\mathcal{VF}} M_\infty$

if $M_j \xrightarrow{\text{SWIF}} M_\infty$ and $\lim_{j \to \infty} \text{Vol}(M_j) = M(M_{\text{SWIF}})$.

**In Lecture 4:** we will learn that $\mathcal{VF}$ convergence implies measured convergence and other strong consequences of this notion that can be proven using Ambrosio-Kirchheim Theory.

The following theorem enables us to prove $\mathcal{VF}$ convergence for sequences of compact oriented Riemannian manifolds:

**Allen-Perales-Sormani:** [arXiv:2003.01172]

\[
M_j \xrightarrow{\text{VADB}} M_\infty \implies M_j \xrightarrow{\mathcal{VF}} M_\infty.
\]

**Defn:** Volume Above Distance Below Conv: $M_j \xrightarrow{\text{VADB}} M_\infty$

if $\text{Vol}(M_j) \to \text{Vol}(M_\infty)$ and $\exists D > 0$ s.t. $\text{Diam}(M_j) \leq D$ and $\exists C^1 \psi_j : M_\infty \to M_j$ s.t. $d_j(\psi_j(p), \psi_j(q)) \geq d_\infty(p, q)$ $\forall p, q \in M_\infty$. 
Defns of $\mathcal{VF}$ and VADB Convergence

Defn: Volume Preserving Intrinsic Flat Conv: $M_j \xrightarrow{\mathcal{VF}} M_\infty$ if $M_j \xrightarrow{\text{SWIF}} M_\infty$ and $\lim_{j \to \infty} \text{Vol}(M_j) = M(M_{\text{SWIF}})$.

In Lecture 4: we will learn that $\mathcal{VF}$ convergence implies measured convergence and other strong consequences of this notion that can be proven using Ambrosio-Kirchheim Theory.

The following theorem enables us to prove $\mathcal{VF}$ convergence for sequences of compact oriented Riemannian manifolds:


$$M_j \xrightarrow{\text{VADB}} M_\infty \implies M_j \xrightarrow{\mathcal{VF}} M_\infty.$$ 

Defn: Volume Above Distance Below Conv: $M_j \xrightarrow{\text{VADB}} M_\infty$ if $\text{Vol}(M_j) \to \text{Vol}(M_\infty)$ and $\exists D > 0$ s.t. $\text{Diam}(M_j) \leq D$ and $\exists C^1 \psi_j : M_\infty \to M_j$ s.t. $d_j(\psi_j(p), \psi_j(q)) \geq d_\infty(p, q) \ \forall p, q \in M_\infty$.

Open Question: If $\text{Scal}_j \geq 0$ and $M_j \xrightarrow{\text{VADB}} M_\infty$ then which properties of nonnegative scalar curvature hold on $M_\infty$?
Review of Lecture I Notions of Convergence

\[ M_j \xrightarrow{C^k} M_\infty \quad \Rightarrow \quad M_j \xrightarrow{C^0} M_\infty \quad \Rightarrow \quad M_j \xrightarrow{\text{Lip}} M_\infty \quad \Rightarrow \quad M_j \xrightarrow{\text{GH}} M_\infty \]
Review of Lecture I Notions of Convergence

\[ M_j \xrightarrow{C^k} M_\infty \implies M_j \xrightarrow{C^0} M_\infty \implies M_j \xrightarrow{\text{Lip}} M_\infty \implies M_j \xrightarrow{\text{GH}} M_\infty \]

Gromov Compactness:

\[ \text{Diam}(M_j) \leq D \text{ and Ricci}_j \geq H \implies \exists M_{j_k} \xrightarrow{\text{GH}} M_{GH} \]
Review of Lecture I Notions of Convergence

\[ M_j \xrightarrow{C^k} M_\infty \implies M_j \xrightarrow{C^0} M_\infty \implies M_j \xrightarrow{\text{Lip}} M_\infty \implies M_j \xrightarrow{\text{GH}} M_\infty \]

Gromov Compactness:

\[ \text{Diam}(M_j) \leq D \text{ and } \text{Ricci}_j \geq H \implies \exists M_{jk} \xrightarrow{\text{GH}} M_{GH} \]

Sormani-Wenger Compactness:

\[ M_j \xrightarrow{\text{GH}} M_{GH} \text{ and } \text{Vol}(M_j) \leq V \implies \exists M_{jk} \xrightarrow{\text{SWIF}} M_{SWIF} \subset M_{GH} \]
Review of Lecture I Notions of Convergence

\[ M_j \xrightarrow{C^k} M_\infty \implies M_j \xrightarrow{C^0} M_\infty \implies M_j \xrightarrow{\text{Lip}} M_\infty \implies M_j \xrightarrow{\text{GH}} M_\infty \]

**Gromov Compactness:**
\[ \text{Diam}(M_j) \leq D \text{ and } \text{Ricci}_j \geq H \implies \exists M_{jk} \xrightarrow{\text{GH}} M_{GH} \]

**Sormani-Wenger Compactness:**
\[ M_j \xrightarrow{\text{GH}} M_{GH} \text{ and } \text{Vol}(M_j) \leq V \implies \exists M_{jk} \xrightarrow{\text{SWIF}} M_{SWIF} \subset M_{GH} \]

**Sormani-Wenger, Matveev-Portegies:**
\[ \text{Vol}(M_j) \geq V_1 \text{ and } \text{Ricci}_j \geq H \implies M_{SWIF} = M_{GH} \]
Review of Lecture I Notions of Convergence

\[ M_j \xrightarrow{C^k} M_\infty \quad \Rightarrow \quad M_j \xrightarrow{C^0} M_\infty \quad \Rightarrow \quad M_j \xrightarrow{\text{Lip}} M_\infty \quad \Rightarrow \quad M_j \xrightarrow{\text{GH}} M_\infty \]

**Gromov Compactness:**
\[ \text{Diam}(M_j) \leq D \text{ and } \text{Ricci}_j \geq H \quad \Rightarrow \quad \exists M_{j_k} \xrightarrow{\text{GH}} M_{GH} \]

**Sormani-Wenger Compactness:**
\[ M_j \xrightarrow{\text{GH}} M_{GH} \text{ and } \text{Vol}(M_j) \leq V \quad \Rightarrow \quad \exists M_{j_k} \xrightarrow{\text{SWIF}} M_{SWIF} \subset M_{GH} \]

**Sormani-Wenger, Matveev-Portegies:**
\[ \text{Vol}(M_j) \geq V_1 \text{ and } \text{Ricci}_j \geq H \quad \Rightarrow \quad M_{SWIF} = M_{GH} \]

**Wenger Compactness:**
\[ \text{Diam}(M_j) \leq D \text{ and } \text{Vol}(M_j) \leq V \quad \Rightarrow \quad \exists M_{j_k} \xrightarrow{\text{SWIF}} M_{SWIF} \]
Review of Lecture I Notions of Convergence

\[ M_j \xrightarrow{C^k} M_\infty \implies M_j \xrightarrow{C^0} M_\infty \implies M_j \xrightarrow{\text{Lip}} M_\infty \implies M_j \xrightarrow{\text{GH}} M_\infty \]

Gromov Compactness:

\[ \text{Diam}(M_j) \leq D \text{ and Ricci}_j \geq H \implies \exists M_{j_k} \xrightarrow{\text{GH}} M_{GH} \]

Sormani-Wenger Compactness:

\[ M_j \xrightarrow{\text{GH}} M_{GH} \text{ and Vol}(M_j) \leq V \implies \exists M_{j_k} \xrightarrow{\text{SWIF}} M_{SWIF} \subset M_{GH} \]

Sormani-Wenger, Matveev-Portegies:

\[ \text{Vol}(M_j) \geq V_1 \text{ and Ricci}_j \geq H \implies M_{SWIF} = M_{GH} \]

Wenger Compactness:

\[ \text{Diam}(M_j) \leq D \text{ and Vol}(M_j) \leq V \implies \exists M_{j_k} \xrightarrow{\text{SWIF}} M_{SWIF} \]

Allen-Perales-Sormani:

\[ M_j \xrightarrow{\text{Lip}} M \implies M_j \xrightarrow{\text{VADB}} M \implies M_j \xrightarrow{\mathcal{V}\mathcal{F}} M \implies M_j \xrightarrow{\text{SWIF}} M \]
Review of Lecture I Notions of Convergence

\[ M_j \xrightarrow{C^k} M_\infty \implies M_j \xrightarrow{C^0} M_\infty \implies M_j \xrightarrow{\text{Lip}} M_\infty \implies M_j \xrightarrow{\text{GH}} M_\infty \]

Gromov Compactness:

\[ \text{Diam}(M_j) \leq D \text{ and Ricci}_j \geq H \implies \exists M_{j_k} \xrightarrow{\text{GH}} M_{GH} \]

Sormani-Wenger Compactness:

\[ M_j \xrightarrow{\text{GH}} M_{GH} \text{ and } \text{Vol}(M_j) \leq V \implies \exists M_{j_k} \xrightarrow{\text{SWIF}} M_{SWIF} \subset M_{GH} \]

Sormani-Wenger, Matveev-Portegies:

\[ \text{Vol}(M_j) \geq V_1 \text{ and Ricci}_j \geq H \implies M_{SWIF} = M_{GH} \]

Wenger Compactness:

\[ \text{Diam}(M_j) \leq D \text{ and } \text{Vol}(M_j) \leq V \implies \exists M_{j_k} \xrightarrow{\text{SWIF}} M_{SWIF} \]

Allen-Perales-Sormani:

\[ M_j \xrightarrow{\text{Lip}} M \implies M_j \xrightarrow{\text{VADB}} M \implies M_j \xrightarrow{\mathcal{V}_\mathcal{F}} M \implies M_j \xrightarrow{\text{SWIF}} M \]

Gromov-Sormani IAS Scalar Compactness Conjecture:

\[ \text{Diam}_j \leq D, \text{ Vol}_j \leq V, \text{ Scal}_j \geq 0, \text{ MinA}_j \geq A \implies \exists M_{j_k} \xrightarrow{\mathcal{V}_\mathcal{F}} M_{SWIF} \]
Lecture 1: Geometric Notions of Convergence DONE!

\[ M_j \xrightarrow{C^k} M_\infty \quad \Rightarrow \quad M_j \xrightarrow{C^0} M_\infty \quad \Rightarrow \quad M_j \xrightarrow{\text{Lip}} M_\infty \quad \Rightarrow \quad M_j \xrightarrow{\text{GH}} M_\infty \]

\[ M_j \xrightarrow{\text{Lip}} M \quad \Rightarrow \quad M_j \xrightarrow{\text{VADB}} M \quad \Rightarrow \quad M_j \xrightarrow{\mathcal{VF}} M \quad \Rightarrow \quad M_j \xrightarrow{\text{SWIF}} M \]
Intrinsic Flat and Gromov-Hausdorff Convergence

Lecture 1: Geometric Notions of Convergence DONE!

\[ M_j \overset{C^k}{\longrightarrow} M_\infty \implies M_j \overset{C^0}{\longrightarrow} M_\infty \implies M_j \overset{\text{Lip}}{\longrightarrow} M_\infty \implies M_j \overset{\text{GH}}{\longrightarrow} M_\infty \]

\[ M_j \overset{\text{Lip}}{\longrightarrow} M \implies M_j \overset{\text{VADB}}{\longrightarrow} M \implies M_j \overset{\nu \mathcal{F}}{\longrightarrow} M \implies M_j \overset{\text{SWIF}}{\longrightarrow} M \]

[Gromov Structures-Metriques] [Burago-Burago-Ivanov Text] [Sormani-Wenger JDG-2011] [Allen-Perales-Sormani VADB] [Conjectures on Convergence and Scalar arXiv: 2103.10093]
Intrinsic Flat and Gromov-Hausdorff Convergence

Lecture 1: Geometric Notions of Convergence DONE!

\[ M_j \xrightarrow{C^k} M_\infty \quad \Rightarrow \quad M_j \xrightarrow{C^0} M_\infty \quad \Rightarrow \quad M_j \xrightarrow{\text{Lip}} M_\infty \quad \Rightarrow \quad M_j \xrightarrow{\text{GH}} M_\infty \]

\[ M_j \xrightarrow{\text{Lip}} M \quad \Rightarrow \quad M_j \xrightarrow{\text{VADB}} M \quad \Rightarrow \quad M_j \xrightarrow{\text{VF}} M \quad \Rightarrow \quad M_j \xrightarrow{\text{SWIF}} M \]

[Gromov Structures-Metriques] [Burago-Burago-Ivanov Text]
[Sormani-Wenger JDG-2011] [Allen-Perales-Sormani VADB]
[Conjectures on Convergence and Scalar arXiv: 2103.10093]

Lecture 2: Open Problems about Scalar Curvature NEXT!
Lecture 1: Geometric Notions of Convergence DONE!

\[ M_j \xrightarrow{C^k} M_\infty \implies M_j \xrightarrow{C^0} M_\infty \implies M_j \xrightarrow{\text{Lip}} M_\infty \implies M_j \xrightarrow{\text{GH}} M_\infty \]

\[ M_j \xrightarrow{\text{Lip}} M \implies M_j \xrightarrow{\text{VADB}} M \implies M_j \xrightarrow{\mathcal{VF}} M \implies M_j \xrightarrow{\text{SWIF}} M \]

[Gromov Structures-Metriques] [Burago-Burago-Ivanov Text]
[Sormani-Wenger JDG-2011] [Allen-Perales-Sormani VADB]
[Conjectures on Convergence and Scalar arXiv: 2103.10093]

Lecture 2: Open Problems about Scalar Curvature NEXT!

\mathcal{VF} Scalar Compactness Conjecture
Intrinsic Flat and Gromov-Hausdorff Convergence

Lecture 1: Geometric Notions of Convergence DONE!

\[ M_j \xrightarrow{C^k} M_\infty \implies M_j \xrightarrow{C^0} M_\infty \implies M_j \xrightarrow{\text{Lip}} M_\infty \implies M_j \xrightarrow{\text{GH}} M_\infty \]

\[ M_j \xrightarrow{\text{Lip}} M \implies M_j \xrightarrow{\text{VADB}} M \implies M_j \xrightarrow{\mathcal{V} \mathcal{F}} M \implies M_j \xrightarrow{\text{SWIF}} M \]

[Gromov Structures-Metriques] [Burago-Burago-Ivanov Text]
[Sormani-Wenger JDG-2011] [Allen-Perales-Sormani VADB]
[Conjectures on Convergence and Scalar arXiv: 2103.10093]

Lecture 2: Open Problems about Scalar Curvature NEXT!

\( \mathcal{V} \mathcal{F} \) Scalar Compactness Conjecture
\( \mathcal{V} \mathcal{F} \) Almost Rigidity of the Positive Mass Theorem
Intrinsic Flat and Gromov-Hausdorff Convergence

Lecture 1: Geometric Notions of Convergence DONE!

\[ M_j \xrightarrow{C^k} M_\infty \quad \Rightarrow \quad M_j \xrightarrow{C^0} M_\infty \quad \Rightarrow \quad M_j \xrightarrow{\text{Lip}} M_\infty \quad \Rightarrow \quad M_j \xrightarrow{\text{GH}} M_\infty \]

\[ M_j \xrightarrow{\text{Lip}} M \quad \Rightarrow \quad M_j \xrightarrow{\text{VADB}} M \quad \Rightarrow \quad M_j \xrightarrow{\mathcal{V}_F} M \quad \Rightarrow \quad M_j \xrightarrow{\text{SWIF}} M \]

[Gromov Structures-Metriques] [Burago-Burago-Ivanov Text]
[Sormani-Wenger JDG-2011] [Allen-Perales-Sormani VADB]
[Conjectures on Convergence and Scalar arXiv: 2103.10093]

Lecture 2: Open Problems about Scalar Curvature NEXT!

\( \mathcal{V}_F \) Scalar Compactness Conjecture
\( \mathcal{V}_F \) Almost Rigidity of the Positive Mass Theorem
\( \mathcal{V}_F \) Geometric Stability of the Scalar Torus Rigidity and
Intrinsic Flat and Gromov-Hausdorff Convergence

**Lecture 1: Geometric Notions of Convergence** DONE!

\[ M_j \overset{C^k}{\rightarrow} M_\infty \implies M_j \overset{C^0}{\rightarrow} M_\infty \implies M_j \overset{\text{Lip}}{\rightarrow} M_\infty \implies M_j \overset{\text{GH}}{\rightarrow} M_\infty \]

\[ M_j \overset{\text{Lip}}{\rightarrow} M \implies M_j \overset{\text{VADB}}{\rightarrow} M \implies M_j \overset{\mathcal{VF}}{\rightarrow} M \implies M_j \overset{\text{SWIF}}{\rightarrow} M \]

[Gromov Structures-Metriques] [Burago-Burago-Ivanov Text]
[Sormani-Wenger JDG-2011] [Allen-Perales-Sormani VADB]
[Conjectures on Convergence and Scalar arXiv: 2103.10093]

**Lecture 2: Open Problems about Scalar Curvature** NEXT!

\( \mathcal{VF} \) Scalar Compactness Conjecture

\( \mathcal{VF} \) Almost Rigidity of the Positive Mass Theorem

\( \mathcal{VF} \) Geometric Stability of the Scalar Torus Rigidity and Scalar Sphere Rigidity and Prism Rigidity Theorems
Intrinsic Flat and Gromov-Hausdorff Convergence

Lecture 1: Geometric Notions of Convergence DONE!

\[ M_j \xrightarrow{C^k} M_\infty \implies M_j \xrightarrow{C^0} M_\infty \implies M_j \xrightarrow{\text{Lip}} M_\infty \implies M_j \xrightarrow{\text{GH}} M_\infty \]

\[ M_j \xrightarrow{\text{Lip}} M \implies M_j \xrightarrow{\text{VADB}} M \implies M_j \xrightarrow{\mathcal{VF}} M \implies M_j \xrightarrow{\text{SWIF}} M \]

[Gromov Structures-Metriques] [Burago-Burago-Ivanov Text]
[Sormani-Wenger JDG-2011] [Allen-Perales-Sormani VADB]
[Conjectures on Convergence and Scalar arXiv: 2103.10093]

Lecture 2: Open Problems about Scalar Curvature NEXT!

\mathcal{VF} Scalar Compactness Conjecture
\mathcal{VF} Almost Rigidity of the Positive Mass Theorem
\mathcal{VF} Geometric Stability of the Scalar Torus Rigidity and
Scalar Sphere Rigidity and Prism Rigidity Theorems
Special Cases involving VADB and Volume Convergence
Intrinsic Flat and Gromov-Hausdorff Convergence

Lecture 1: Geometric Notions of Convergence DONE!

\[ M_j \xrightarrow{C^k} M_\infty \implies M_j \xrightarrow{C^0} M_\infty \implies M_j \xrightarrow{\text{Lip}} M_\infty \implies M_j \xrightarrow{\text{GH}} M_\infty \]

\[ M_j \xrightarrow{\text{Lip}} M \implies M_j \xrightarrow{\text{VADB}} M \implies M_j \xrightarrow{\mathcal{VF}} M \implies M_j \xrightarrow{\text{SWIF}} M \]

[Gromov Structures-Metriques] [Burago-Burago-Ivanov Text]
[Sormani-Wenger JDG-2011] [Allen-Perales-Sormani VADB]
[Conjectures on Convergence and Scalar arXiv: 2103.10093]

Lecture 2: Open Problems about Scalar Curvature NEXT!

\( \mathcal{VF} \) Scalar Compactness Conjecture
\( \mathcal{VF} \) Almost Rigidity of the Positive Mass Theorem
\( \mathcal{VF} \) Geometric Stability of the Scalar Torus Rigidity and
Scalar Sphere Rigidity and Prism Rigidity Theorems

Special Cases involving VADB and Volume Convergence
will be assigned to teams of interested students/postdocs
[Conjectures on Convergence and Scalar arXiv: 2103.10093]

Other papers on SWIF convergence may be found at: https://sites.google.com/site/intrinsicflatconvergence/

This talk may be downloaded at my website: https://sites.google.com/site/professorsormani/

Feel free to email me with questions: sormanic@gmail.com

Thank you for listening - Christina Sormani