Intrinsic Flat and Gromov-Hausdorff Convergence

Christina Sormani

CUNY GC and Lehman College

Lectures III-IV: Proving Intrinsic Flat Convergence
Intrinsic Flat and Gromov-Hausdorff Convergence

**Lecture 1: Geometric Notions of Convergence DONE!**

- Reviewed $C^k$, $C^0$, Lip, and GH Convergence,
- Sormani-Wenger Intrinsic Flat Convergence (SWIF) or ($\mathcal{F}$),
- Volume Preserving Intrinsic Flat Convergence ($\mathcal{VF}$)
- Allen-Perales-Sormani (VADB) Convergence
Lecture 1: Geometric Notions of Convergence DONE!
Reviewed $C^k$, $C^0$, Lip, and GH Convergence,
Sormani-Wenger Intrinsic Flat Convergence ($\mathcal{F}$) or ($\mathcal{F}$),
Volume Preserving Intrinsic Flat Convergence ($\mathcal{VF}$)
Allen-Perales-Sormani (VADB) Convergence

Lecture 2: Open Problems on Scalar Curvature DONE!
Consider: Three Dimensional Manifolds $M_j^3$ with $\text{Scal} \geq H$
and their Limit Spaces $M_\infty$
Which Geometric Properties of $M_j^3$ with $\text{Scal} \geq H$
persist on their Limit Spaces $M_\infty$?
Which Rigidity Theorems for $M^3$ with $\text{Scal} \geq H$
have Geometric Stability?
[Conjectures on Conv and Scalar (2021) arXiv: 2103.10093]
Intrinsic Flat and Gromov-Hausdorff Convergence

Lecture 1: Geometric Notions of Convergence DONE!
Reviewed $C^k$, $C^0$, Lip, and GH Convergence,
Sormani-Wenger Intrinsic Flat Convergence (SWIF) or ($\mathcal{F}$),
Volume Preserving Intrinsic Flat Convergence ($\mathcal{VF}$)
Allen-Perales-Sormani (VADB) Convergence

Lecture 2: Open Problems on Scalar Curvature DONE!
Consider: Three Dimensional Manifolds $M^3_j$ with $\text{Scal} \geq H$
and their Limit Spaces $M^\infty$
Which Geometric Properties of $M^3_j$ with $\text{Scal} \geq H$
persist on their Limit Spaces $M^\infty$?
Which Rigidity Theorems for $M^3$ with $\text{Scal} \geq H$
have Geometric Stability?
[Conjectures on Conv and Scalar (2021) arXiv: 2103.10093]

Lectures 3&4: Techniques to Apply to Prove Convergence
See https://sites.google.com/site/intrinsicflatconvergence/
Volume Preserving Intrinsic Flat $\mathcal{VF}$ Convergence

Defn: $M_j \xrightarrow{VF} M_\infty$ if $d(F(M_j, M_\infty)) \rightarrow 0$: 

Defn: $M_j \xrightarrow{F} M_\infty$ if $d(SWIF)(M_j, M_\infty) \rightarrow 0$: 
Volume Preserving Intrinsic Flat $\mathcal{VF}$ Convergence

Defn: $M_j \xrightarrow{\mathcal{VF}} M_\infty$ if $M_j \xrightarrow{\mathcal{F}} M_\infty$ and $\text{Vol}(M_j) \rightarrow \text{Vol}(M_\infty)$. 
Volume Preserving Intrinsic Flat $\mathcal{VF}$ Convergence

Defn: $M_j \xrightarrow{\mathcal{VF}} M_\infty$ if $M_j \xrightarrow{\mathcal{F}} M_\infty$ and $\text{Vol}(M_j) \to \text{Vol}(M_\infty)$.

Defn: $M_j \xrightarrow{\mathcal{F}} M_\infty$ if $d_{\mathcal{F}}(M_j, M_\infty) = d_{SWIF}(M_j, M_\infty) \to 0$.

Sormani-Wenger: Intrinsic Flat Distance

The intrinsic flat distance between oriented manifolds $M_i^m$ is:

$$d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d_{\mathcal{F}}^Z (\varphi_1#[[M_1^m]], \varphi_2#[[M_2^m]]) \mid \varphi_i : M_i^m \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i^m \to Z$.

Here: $d_{\mathcal{F}}^Z (\varphi_1#[[M_1^m]], \varphi_2#[[M_2^m]])$ is the Federer-Fleming Flat dist

$$= \inf \left\{ \text{M}_{\text{area}}(A) + \text{M}_{\text{vol}}(B) : A + \partial B = \varphi_1#[[M_1^m]] - \varphi_2#[[M_2^m]] \right\}$$
History of the Flat Norm in Euclidean Space:

**Whitney (1957):** Flat norm between submanifolds $N_i$ in $\mathbb{R}^N$:

\[ |N_1 - N_2|_b = \inf \left\{ M(\textcolor{green}{A}) + M(\textcolor{yellow}{B}) \right\} \]

where \textcolor{green}{A} and \textcolor{yellow}{B} are chains

such that \textcolor{green}{A} + \partial \textcolor{yellow}{B} = N_1 - N_2.
History of the Flat Norm in Euclidean Space:

**Whitney (1957):** Flat norm between submanifolds $N_i$ in $\mathbb{R}^N$:

$$|N_1 - N_2|_b = \inf \left\{ M(A) + M(B) \right\}$$

where $A$ and $B$ are chains such that $A + \partial B = N_1 - N_2$.

**Federer-Fleming (1959):** Use Whitney’s definition but now $A$ and $B$ are currents acting on differential forms, $\omega$, in $\mathbb{R}^N$. They view a submanifold $\varphi(M^m)$ as an $m$-current $\varphi\#[[M^m]]$:

$$\varphi\#[[M^m]](\omega) = \int_{\varphi(M^m)} \omega = \int_{M^m} \varphi^*\omega$$

where $\omega$ is an $m$-form.

$$\varphi\#[[M]](f \, d\pi_1 \wedge \cdots \wedge d\pi_m) =$$
History of the Flat Norm in Euclidean Space:

**Whitney (1957):** Flat norm between submanifolds $N_i$ in $\mathbb{R}^N$:

$$|N_1 - N_2|_b = \inf \left\{ M(A) + M(B) \right\}$$

where $A$ and $B$ are chains such that $A + \partial B = N_1 - N_2$.

**Federer-Fleming (1959):** Use Whitney's definition but now $A$ and $B$ are currents acting on differential forms, $\omega$, in $\mathbb{R}^N$. They view a submanifold $\varphi(M^m)$ as an $m$-current $\varphi\#[[M^m]]$:

$$\varphi\#[[M^m]](\omega) = \int_{\varphi(M^m)} \omega = \int_{M^m} \varphi^*\omega$$

where $\omega$ is an $m$-form.

$$\varphi\#[[M]](f \, d\pi_1 \wedge \cdots \wedge d\pi_m) = \int_M$$
History of the Flat Norm in Euclidean Space:

**Whitney (1957):** Flat norm between submanifolds $N_i$ in $\mathbb{R}^N$:

$$|N_1 - N_2|_b = \inf \left\{ M(A) + M(B) \right\}$$

where $A$ and $B$ are chains such that $A + \partial B = N_1 - N_2$.

**Federer-Fleming (1959):** Use Whitney’s definition but now $A$ and $B$ are currents acting on differential forms, $\omega$, in $\mathbb{R}^N$. They view a submanifold $\varphi(M^m)$ as an $m$-current $\varphi#[[M^m]]$:

$$\varphi#[[M^m]](\omega) = \int_{\varphi(M^m)} \omega = \int_{M^m} \varphi^*\omega$$

where $\omega$ is an $m$-form.

$$\varphi#[[M]](f \, d\pi_1 \wedge \cdots \wedge d\pi_m) = \int_M (f \circ \varphi)$$
History of the Flat Norm in Euclidean Space:

**Whitney (1957):** Flat norm between submanifolds $N_i$ in $\mathbb{R}^N$:

$$|N_1 - N_2|_b = \inf \left\{ \mathbf{M}(A) + \mathbf{M}(B) \right\}$$

where $A$ and $B$ are chains such that $A + \partial B = N_1 - N_2$.

**Federer-Fleming (1959):** Use Whitney’s definition but now $A$ and $B$ are currents acting on differential forms, $\omega$, in $\mathbb{R}^N$. They view a submanifold $\varphi(M^m)$ as an $m$-current $\varphi\#[[M^m]]$:

$$\varphi\#[[M^m]](\omega) = \int_{\varphi(M^m)} \omega = \int_{M^m} \varphi^*\omega$$

where $\omega$ is an $m$-form.

$$\varphi\#[[M]](f d\pi_1 \wedge \cdots \wedge d\pi_m) = \int_M (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi).$$
History of the Flat Norm in Euclidean Space:

**Whitney (1957):** Flat norm between submanifolds $N_i$ in $\mathbb{R}^N$:

$$|N_1 - N_2|_b = \inf \left\{ M(A) + M(B) \right\}$$

where $A$ and $B$ are chains such that $A + \partial B = N_1 - N_2$.

**Federer-Fleming (1959):** Use Whitney’s definition but now $A$ and $B$ are currents acting on differential forms, $\omega$, in $\mathbb{R}^N$. They view a submanifold $\varphi(M^m)$ as an $m$-current $\varphi\#[[M^m]]$:

$$\varphi\#[[M^m]](\omega) = \int_{\varphi(M^m)} \omega = \int_{M^m} \varphi^* \omega \text{ where } \omega \text{ is an } m\text{-form.}$$

$$\varphi\#[[M]](f \ d\pi_1 \wedge \cdots \wedge d\pi_m) = \int_M (f \circ \varphi) \ d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi).$$

Given a current $T$ they define $\partial T : \partial T(\omega) = T(d\omega)$ so that:
History of the Flat Norm in Euclidean Space:

**Whitney (1957):** Flat norm between submanifolds $N_i$ in $\mathbb{R}^N$:

$$|N_1 - N_2|_b = \inf \left\{ M(A) + M(B) \right\}$$

where $A$ and $B$ are chains such that $A + \partial B = N_1 - N_2$.

**Federer-Fleming (1959):** Use Whitney’s definition but now $A$ and $B$ are currents acting on differential forms, $\omega$, in $\mathbb{R}^N$. They view a submanifold $\varphi(M^m)$ as an $m$-current $\varphi\#[[M^m]]$:

$$\varphi\#[[M^m]](\omega) = \int_{\varphi(M^m)} \omega = \int_{M^m} \varphi^* \omega$$

where $\omega$ is an $m$-form.

$$\varphi\#[[M]](f d\pi_1 \wedge \cdots \wedge d\pi_m) = \int_M (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi).$$

Given a current $T$ they define $\partial T : \partial T(\omega) = T(d\omega)$ so that:

$$\partial[[N^m]](\omega) = [[N^m]](d\omega) = \int_N d\omega = \int_{\partial N} \omega = [[\partial N^m]](\omega)$$
History of the Flat Norm in Euclidean Space:

**Whitney (1957):** Flat norm between submanifolds $N_i$ in $\mathbb{R}^N$:

$$ |N_1 - N_2|_b = \inf \left\{ M(A) + M(B) \right\}$$

where $A$ and $B$ are chains such that $A + \partial B = N_1 - N_2$.

**Federer-Fleming (1959):** Use Whitney’s definition but now $A$ and $B$ are currents acting on differential forms, $\omega$, in $\mathbb{R}^N$. They view a submanifold $\varphi(M^m)$ as an $m$-current $\varphi\#[[M^m]]$:

$$ \varphi\#[[M^m]](\omega) = \int_{\varphi(M^m)} \omega = \int_{M^m} \varphi^*\omega \text{ where } \omega \text{ is an } m\text{-form.}$$

$$ \varphi\#[[M]](f \, d\pi_1 \wedge \cdots \wedge d\pi_m) = \int_M (f \circ \varphi) \, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi).$$

Given a current $T$ they define $\partial T : \partial T(\omega) = T(d\omega)$ so that:

$$ \partial[[N^m]](\omega) = [[N^m]](d\omega) = \int_N d\omega = \int_{\partial N} \omega = [[\partial N^m]](\omega)$$

where $d(f \, d\pi_1 \wedge \cdots \wedge d\pi_m) = df \wedge d\pi_1 \wedge \cdots \wedge d\pi_m$. 
History of the Flat Norm in Euclidean Space:

**Whitney (1957):** Flat norm in $\mathbb{R}^N$:

$$|N_1 - N_2|_b = \inf \left\{ M(A) + M(B) \right\}$$

such that $A + \partial B = N_1 - N_2$. “

**Federer-Fleming (1959):** Use Whitney's definition but now $A$ and $B$ are integral currents acting on differential forms, $\omega$, in $\mathbb{R}^N$.

Defn: an integral current, $T$, is an integer rectifiable current whose boundary $\partial T$ defined by $\partial T(\omega) = T(d\omega)$ is also integer rectifiable.

where an integer rectifiable current, $T$, has a countable collection of pairwise disjoint biLip charts $\phi_i: A_i \to \phi_i(A_i) \subset \mathbb{R}^N$ and weights $a_i \in \mathbb{Z}$ such that $T(\omega) = \sum_{i=1}^{\infty} a_i \int_{A_i} \phi_i^* \omega$.

Defn: The mass $M(T) = ||T||(\mathbb{R}^N) = \sum_{i=1}^{\infty} |a_i| H^m(\phi_i(A_i))$.

Compactness Thm [FF]: If integral currents $T_j$ have $\text{spt}(T_j) \subset K$ compact, and $M(T_j) \leq V$ and $M(\partial T_j) \leq A$ then $\exists$ subseq $T_{j_k}$ and an integral current $T_\infty$ s.t. $||T_{j_k} - T_\infty|| \to 0$.

Furthermore: $\partial T_{j_k} \to \partial T_\infty$ and $\lim inf_{j \to \infty} M(T_j) \geq M(T_\infty)$ and $T_j(\omega) \to T_\infty(\omega)$ for any diff form $\omega$. 
History of the Flat Norm in Euclidean Space:

**Whitney (1957):** Flat norm in $\mathbb{R}^N$:

$$|N_1 - N_2|_b = \inf \left\{ M(A) + M(B) \right\}$$

such that $A + \partial B = N_1 - N_2$.

**Federer-Fleming (1959):** Use Whitney’s definition but now $A$ and $B$ are integral currents acting on diff forms, $\omega$, in $\mathbb{R}^N$. 

History of the Flat Norm in Euclidean Space:

**Whitney (1957):** Flat norm in $\mathbb{R}^N$:

$$|N_1 - N_2|_b = \inf \left\{ M(A) + M(B) \right\}$$

such that $A + \partial B = N_1 - N_2$.

**Federer-Fleming (1959):** Use Whitney’s definition but now $A$ and $B$ are integral currents acting on diff forms, $\omega$, in $\mathbb{R}^N$.

Defn: an integral current, $T$, is an integer rectifiable current whose boundary $\partial T$ defined by $\partial T(\omega) = T(d\omega)$ is also integer rectifiable.
History of the Flat Norm in Euclidean Space:

**Whitney (1957):** Flat norm in $\mathbb{R}^N$:

\[ |N_1 - N_2| = \inf \left\{ M(A) + M(B) \right\} \]

such that $A + \partial B = N_1 - N_2$.

**Federer-Fleming (1959):** Use Whitney’s definition but now $A$ and $B$ are integral currents acting on diff forms, $\omega$, in $\mathbb{R}^N$.

Defn: an integral current, $T$, is an integer rectifiable current whose boundary $\partial T$ defined by $\partial T(\omega) = T(d\omega)$ is also integer rectifiable. where an integer rectifiable current, $T$, has a countable collection of pairwise disjoint biLip charts $\varphi_i : A_i \to \varphi_i(A_i) \subset \mathbb{R}^N$ and weights $a_i \in \mathbb{Z}$ such that $T(\omega) = \sum_{i=1}^{\infty} a_i \int_{A_i} \varphi_i^* \omega$. 
History of the Flat Norm in Euclidean Space:

**Whitney (1957):** Flat norm in $\mathbb{R}^N$:

$$|\mathcal{N}_1 - \mathcal{N}_2|_b = \inf \left\{ M(\mathcal{A}) + M(\mathcal{B}) \right\}$$

such that $\mathcal{A} + \partial \mathcal{B} = \mathcal{N}_1 - \mathcal{N}_2$.

**Federer-Fleming (1959):** Use Whitney’s definition but now $\mathcal{A}$ and $\mathcal{B}$ are integral currents acting on diff forms, $\omega$, in $\mathbb{R}^N$.

Defn: an integral current, $\mathcal{T}$, is an integer rectifiable current whose boundary $\partial \mathcal{T}$ defined by $\partial \mathcal{T}(\omega) = \mathcal{T}(d\omega)$ is also integer rectifiable, where an integer rectifiable current, $\mathcal{T}$, has a countable collection of pairwise disjoint biLip charts $\varphi_i : A_i \to \varphi_i(A_i) \subset \mathbb{R}^N$ and weights $a_i \in \mathbb{Z}$ such that $\mathcal{T}(\omega) = \sum_{i=1}^{\infty} a_i \int_{A_i} \varphi_i^* \omega$.

Defn: The mass $M(\mathcal{T}) = \| \mathcal{T} \|(\mathbb{R}^N) = \sum_{i=1}^{\infty} |a_i| \mathcal{H}^m(\varphi_i(A_i))$. 
History of the Flat Norm in Euclidean Space:

**Whitney (1957):** Flat norm in $\mathbb{R}^N$:

$$|N_1 - N_2|_b = \inf \left\{ M(A) + M(B) \right\}$$

such that $A + \partial B = N_1 - N_2$.

**Federer-Fleming (1959):** Use Whitney’s definition but now $A$ and $B$ are integral currents acting on diff forms, $\omega$, in $\mathbb{R}^N$.

Defn: an integral current, $T$, is an integer rectifiable current whose boundary $\partial T$ defined by $\partial T(\omega) = T(d\omega)$ is also integer rectifiable. where an integer rectifiable current, $T$, has a countable collection of pairwise disjoint biLip charts $\varphi_i : A_i \to \varphi_i(A_i) \subset \mathbb{R}^N$ and weights $a_i \in \mathbb{Z}$ such that $T(\omega) = \sum_{i=1}^{\infty} a_i \int_{A_i} \varphi_i^* \omega$.

Defn: The mass $M(T) = \|T\|_{\mathbb{R}^N} = \sum_{i=1}^{\infty} |a_i| \mathcal{H}^m(\varphi_i(A_i))$.

**Compactness Thm [FF]:** If integral currents $T_j$ have $\text{spt}(T_j) \subset K$ compact, and $M(T_j) \leq V$ and $M(\partial T_j) \leq A$ then $\exists$ subseq $T_{j_k}$ and an integral current $T_\infty$ s.t. $|T_{j_k} - T_\infty|_b \to 0$. 
History of the Flat Norm in Euclidean Space:

**Whitney (1957):** Flat norm in $\mathbb{R}^N$:

\[
|N_1 - N_2|_b = \inf \left\{ M\left(\begin{array}{c} A \end{array}\right) + M\left(\begin{array}{c} B \end{array}\right) \right\}
\]

such that $A + \partial B = N_1 - N_2$.

**Federer-Fleming (1959):** Use Whitney's definition but now $A$ and $B$ are integral currents acting on diff forms, $\omega$, in $\mathbb{R}^N$.

**Defn:** an integral current, $T$, is an integer rectifiable current whose boundary $\partial T$ defined by $\partial T(\omega) = T(d\omega)$ is also integer rectifiable. where an integer rectifiable current, $T$, has a countable collection of pairwise disjoint biLip charts $\varphi_i : A_i \rightarrow \varphi_i(A_i) \subset \mathbb{R}^N$ and weights $a_i \in \mathbb{Z}$ such that $T(\omega) = \sum_{i=1}^{\infty} a_i \int_{A_i} \varphi_i^* \omega$.

**Defn:** The mass $M(T) = ||T||(\mathbb{R}^N) = \sum_{i=1}^{\infty} |a_i| \mathcal{H}^m(\varphi_i(A_i))$.

**Compactness Thm [FF]:** If integral currents $T_j$ have $spt(T_j) \subset K$ compact, and $M(T_j) \leq V$ and $M(\partial T_j) \leq A$ then $\exists$ subseq $T_{j_k}$ and an integral current $T_\infty$ s.t. $|T_{j_k} - T_\infty|_b \rightarrow 0$.

Furthermore: $\partial T_j \overset{\mathcal{F}}{\rightarrow} \partial T_\infty$
History of the Flat Norm in Euclidean Space:

**Whitney (1957):** Flat norm in \( \mathbb{R}^N \):
\[
|N_1 - N_2|_b = \inf \left\{ M(A) + M(B) \right\}
\]
such that \( A + \partial B = N_1 - N_2 \).

**Federer-Fleming (1959):** Use Whitney’s definition but now \( A \) and \( B \) are integral currents acting on diff forms, \( \omega \), in \( \mathbb{R}^N \).

Defn: an integral current, \( T \), is an integer rectifiable current whose boundary \( \partial T \) defined by \( \partial T(\omega) = T(d\omega) \) is also integer rectifiable. where an integer rectifiable current, \( T \), has a countable collection of pairwise disjoint biLip charts \( \varphi_i : A_i \to \varphi_i(A_i) \subset \mathbb{R}^N \) and weights \( a_i \in \mathbb{Z} \) such that \( T(\omega) = \sum_{i=1}^{\infty} a_i \int_{A_i} \varphi_i^* \omega \).

Defn: The mass \( M(T) = \|T\|_{\mathbb{R}^N} = \sum_{i=1}^{\infty} |a_i| \mathcal{H}^m(\varphi_i(A_i)) \).

**Compactness Thm [FF]:** If integral currents \( T_j \) have \( \text{spt}(T_j) \subset K \) compact, and \( M(T_j) \leq V \) and \( M(\partial T_j) \leq A \) then \( \exists \) subseq \( T_{jk} \) and an integral current \( T_\infty \) s.t. \( |T_{jk} - T_\infty|_b \to 0 \).

Furthermore: \( \partial T_j \overset{\mathcal{F}}{\to} \partial T_\infty \) and \( \liminf_{j \to \infty} M(T_j) \geq M(T_\infty) \).
History of the Flat Norm in Euclidean Space:

**Whitney (1957):** Flat norm in $\mathbb{R}^N$:

$$|N_1 - N_2|_b = \inf \left\{ M(A) + M(B) \right\}$$

such that $A + \partial B = N_1 - N_2$.

**Federer-Fleming (1959):** Use Whitney’s definition but now $A$ and $B$ are integral currents acting on diff forms, $\omega$, in $\mathbb{R}^N$.

Defn: an integral current, $T$, is an integer rectifiable current whose boundary $\partial T$ defined by $\partial T(\omega) = T(d\omega)$ is also integer rectifiable. where an integer rectifiable current, $T$, has a countable collection of pairwise disjoint biLip charts $\varphi_i : A_i \to \varphi_i(A_i) \subset \mathbb{R}^N$ and weights $a_i \in \mathbb{Z}$ such that $T(\omega) = \sum_{i=1}^{\infty} a_i \int_{A_i} \varphi_i^* \omega$.

Defn: The mass $M(T) = ||T||(\mathbb{R}^N) = \sum_{i=1}^{\infty} |a_i| \mathcal{H}^m(\varphi_i(A_i))$.

**Compactness Thm [FF]:** If integral currents $T_j$ have $spt(T_j) \subset K$ compact, and $M(T_j) \leq V$ and $M(\partial T_j) \leq A$ then $\exists$ subseq $T_{j_k}$ and an integral current $T_\infty$ s.t. $|T_{j_k} - T_\infty|_b \to 0$.

Furthermore: $\partial T_j \overset{F}{\to} \partial T_\infty$ and $\lim \inf_{j \to \infty} M(T_j) \geq M(T_\infty)$ and $T_j(\omega) \to T_\infty(\omega)$ for any diff form $\omega$. 
Federer-Fleming (1959): Currents in $\mathbb{R}^N$ act on diff forms. Given a smooth $\varphi : M^m \to \mathbb{R}^N$, define a current acting on forms:

$$\varphi_#[[M]](f \; d\pi_1 \wedge \cdots \wedge d\pi_m) = \int_M (f \circ \varphi) \; d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi).$$
Federer-Fleming (1959): Currents in $\mathbb{R}^N$ act on diff forms. Given a smooth $\varphi : M^m \to \mathbb{R}^N$, define a current acting on forms:

$$\varphi_# [[M]] (f d\pi_1 \wedge \cdots \wedge d\pi_m) = \int_M (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi).$$

Given a current $T$ acting on form $\omega$ define $\partial T : \partial T(\omega) = T(d\omega)$ where

$$d(f d\pi_1 \wedge \cdots \wedge d\pi_m) = df \wedge d\pi_1 \wedge \cdots \wedge d\pi_m.$$
Currents in Metric Spaces:

**Federer-Fleming (1959):** Currents in $\mathbb{R}^N$ act on diff forms. Given a smooth $\varphi : M^m \to \mathbb{R}^N$, define a current acting on forms:

$$\varphi_\#[[M]](f \, d\pi_1 \wedge \cdots \wedge d\pi_m) = \int_M (f \circ \varphi) \, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi).$$

Given a current $T$ acting on form $\omega$ define $\partial T : \partial T(\omega) = T(d\omega)$ where

$$d(f \, d\pi_1 \wedge \cdots \wedge d\pi_m) = df \wedge d\pi_1 \wedge \cdots \wedge d\pi_m.$$

**DeGiorgi (1995):** For a complete metric space, $Z$, 

Federer-Fleming (1959): Currents in $\mathbb{R}^N$ act on diff forms. Given a smooth $\varphi : M^m \to \mathbb{R}^N$, define a current acting on forms:

$$\varphi#[[M]](f \, d\pi_1 \wedge \cdots \wedge d\pi_m) = \int_M (f \circ \varphi) \, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi).$$

Given a current $T$ acting on form $\omega$ define $\partial T : \partial T(\omega) = T(d\omega)$ where $d(f \, d\pi_1 \wedge \cdots \wedge d\pi_m) = df \wedge d\pi_1 \wedge \cdots \wedge d\pi_m$.

DeGiorgi (1995): For a complete metric space, $Z$, replace diff forms $f \, d\pi_1 \wedge \cdots \wedge d\pi_m$ with tuples $(f, \pi_1, \ldots, \pi_m)$ s.t. $f : Z \to \mathbb{R}$ is bounded Lipschitz and $\pi_i : Z \to \mathbb{R}$ are Lipschitz.
Federer-Fleming (1959): Currents in $\mathbb{R}^N$ act on diff forms. Given a smooth $\varphi : M^m \to \mathbb{R}^N$, define a current acting on forms:

$$\varphi^\#[[M]](f \, d\pi_1 \wedge \cdots \wedge d\pi_m) = \int_M (f \circ \varphi) \, d(\pi_1 \circ \varphi)^\wedge \cdots \wedge d(\pi_m \circ \varphi).$$

Given a current $T$ acting on form $\omega$ define $\partial T$:

$$\partial T(\omega) = T(d\omega)$$

where $d(f \, d\pi_1 \wedge \cdots \wedge d\pi_m) = df \wedge d\pi_1 \wedge \cdots \wedge d\pi_m$.

DeGiorgi (1995): For a complete metric space, $Z$, replace diff forms $f \, d\pi_1 \wedge \cdots \wedge d\pi_m$ with tuples $(f, \pi_1, \ldots, \pi_m)$ s.t. $f : Z \to \mathbb{R}$ is bounded Lipschitz and $\pi_i : Z \to \mathbb{R}$ are Lipschitz. Given a Lipschitz $\varphi : M^m \to Z$, define a current acting on tuples:

$$\varphi^\#[[M]](f, \pi_1, \ldots, \pi_m) =$$
**Federer-Fleming (1959):** Currents in $\mathbb{R}^N$ act on diff forms. Given a smooth $\varphi : M^m \to \mathbb{R}^N$, define a current acting on forms:

\[
\varphi_\#[[M]](f \, d\pi_1 \wedge \cdots \wedge d\pi_m) = \int_M (f \circ \varphi) \, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi).
\]

Given a current $T$ acting on form $\omega$ define $\partial T : \partial T(\omega) = T(d\omega)$ where $d(f \, d\pi_1 \wedge \cdots \wedge d\pi_m) = df \wedge d\pi_1 \wedge \cdots \wedge d\pi_m$.

**DeGiorgi (1995):** For a complete metric space, $Z$, replace diff forms $f \, d\pi_1 \wedge \cdots \wedge d\pi_m$ with tuples $(f, \pi_1, \ldots, \pi_m)$ s.t. $f : Z \to \mathbb{R}$ is bounded Lipschitz and $\pi_i : Z \to \mathbb{R}$ are Lipschitz. Given a Lipschitz $\varphi : M^m \to Z$, define a current acting on tuples:

\[
\varphi_\#[[M]](f, \pi_1, \ldots, \pi_m) = \int_M (f \circ \varphi) \, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)
\]
Federer-Fleming (1959): Currents in $\mathbb{R}^N$ act on diff forms. Given a smooth $\varphi: M^m \to \mathbb{R}^N$, define a current acting on forms:

$$\varphi#[[M]](f \, d\pi_1 \wedge \cdots \wedge d\pi_m) = \int_M (f \circ \varphi) \, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi).$$

Given a current $T$ acting on form $\omega$ define $\partial T : \partial T(\omega) = T(d\omega)$ where $d(f \, d\pi_1 \wedge \cdots \wedge d\pi_m) = df \wedge d\pi_1 \wedge \cdots \wedge d\pi_m$.

DeGiorgi (1995): For a complete metric space, $Z$, replace diff forms $f \, d\pi_1 \wedge \cdots \wedge d\pi_m$ with tuples $(f, \pi_1, ..., \pi_m)$ s.t. $f: Z \to \mathbb{R}$ is bounded Lipschitz and $\pi_i: Z \to \mathbb{R}$ are Lipschitz. Given a Lipschitz $\varphi: M^m \to Z$, define a current acting on tuples:

$$\varphi#[[M]](f, \pi_1, ..., \pi_m) = \int_M (f \circ \varphi) \, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

Given a current $T$ acting on a tuple $\omega$ define $\partial T(\omega) = T(d\omega)$ where $d(f, \pi_1, ..., \pi_m) =$.
Federer-Fleming (1959): Currents in $\mathbb{R}^N$ act on diff forms. Given a smooth $\varphi : M^m \to \mathbb{R}^N$, define a current acting on forms:

$$\varphi_#(\pi_1 \cdots \pi_m) = \int_M \varphi \circ \varphi_1 \wedge \cdots \wedge \varphi_m.$$

Given a current $T$ acting on form $\omega$, define $\partial T : \partial T(\omega) = T(d\omega)$
where $d(f \pi_1 \wedge \cdots \wedge d\pi_m) = df \wedge \pi_1 \wedge \cdots \wedge d\pi_m$.

DeGiorgi (1995): For a complete metric space, $Z$, replace diff forms $f \pi_1 \wedge \cdots \wedge d\pi_m$ with tuples $(f, \pi_1, \ldots, \pi_m)$ s.t. $f : Z \to \mathbb{R}$ is bounded Lipschitz and $\pi_i : Z \to \mathbb{R}$ are Lipschitz. Given a Lipschitz $\varphi : M^m \to Z$, define a current acting on tuples:

$$\varphi_#([M])(f, \pi_1, \ldots, \pi_m) = \int_M (f \circ \varphi) \varphi_1 \wedge \cdots \wedge \varphi_m.$$

Given a current $T$ acting on a tuple $\omega$, define $\partial T(\omega) = T(d\omega)$
where $d(f, \pi_1, \ldots, \pi_m) = (1, f, \pi_1, \ldots, \pi_m)$. 

Currents in Metric Spaces:
Integral Currents in Metric Spaces:

**DeGiorgi (1995):** For a complete metric space, $Z$, replace diff forms $f \, d\pi_1 \wedge \cdots \wedge d\pi_m$ with tuples $(f, \pi_1, \ldots, \pi_m)$ s.t. $f : Z \to \mathbb{R}$ is bounded Lipschitz and $\pi_i : Z \to \mathbb{R}$ are Lipschitz. Given a Lipschitz $\varphi : M^m \to Z$, define a current acting on tuples:

$$\varphi \#[[M]](f, \pi_1, \ldots, \pi_m) = \int_M (f \circ \varphi) \, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

Given a current $T$ acting on a tuple $\omega$ define $\partial T(\omega) = T(d\omega)$ where $d(f, \pi_1, \ldots, \pi_m) = (1, f, \pi_1, \ldots, \pi_m)$.
Integral Currents in Metric Spaces:

DeGiorgi (1995): For a complete metric space, \( Z \), replace diff forms \( f \, d\pi_1 \wedge \cdots \wedge d\pi_m \) with tuples \((f, \pi_1, \ldots, \pi_m)\) s.t. \( f : Z \to \mathbb{R} \) is bounded Lipschitz and \( \pi_i : Z \to \mathbb{R} \) are Lipschitz. Given a Lipschitz \( \varphi : M^m \to Z \), define a current acting on tuples:

\[
\varphi \#[[M]](f, \pi_1, \ldots, \pi_m) = \int_M (f \circ \varphi) \, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)
\]

Given a current \( T \) acting on a tuple \( \omega \) define \( \partial T(\omega) = T(d\omega) \) where \( d(f, \pi_1, \ldots, \pi_m) = (1, f, \pi_1, ..., \pi_m) \).

Ambrosio-Kirchheim (2000): an integral current, \( T \), is an integer rectifiable current s.t. \( \partial T \) is also integer rectifiable.
Integral Currents in Metric Spaces:

DeGiorgi (1995): For a complete metric space, $Z$, replace diff forms $f \, d\pi_1 \wedge \cdots \wedge d\pi_m$ with tuples $(f, \pi_1, \ldots, \pi_m)$ s.t. $f : Z \to \mathbb{R}$ is bounded Lipschitz and $\pi_i : Z \to \mathbb{R}$ are Lipschitz. Given a Lipschitz $\varphi : M^m \to Z$, define a current acting on tuples:

$$\varphi^\#[[M]](f, \pi_1, \ldots, \pi_m) = \int_M (f \circ \varphi) \, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

Given a current $T$ acting on a tuple $\omega$ define $\partial T(\omega) = T(d\omega)$ where $d(f, \pi_1, \ldots, \pi_m) = (1, f, \pi_1, \ldots, \pi_m)$.

Ambrosio-Kirchheim (2000): an integral current, $T$, is an integer rectifiable current s.t. $\partial T$ is also integer rectifiable. where an integer rectifiable current, $T$, has cntbly many pairwise disjoint biLip charts $\varphi_i : A_i \to \varphi_i(A_i) \subset Z$ and weights $a_i \in \mathbb{Z}$ s.t.

$$T(f, \pi_1, \ldots, \pi_m) = \sum_{i=1}^\infty a_i \int_{A_i} (f \circ \varphi) \, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

Mass is not the weighted volume in Ambrosio-Kirchheim Theory!
Integral Currents in Metric Spaces:

**DeGiorgi (1995):** For a complete metric space, $Z$, replace diff forms $f \, d\pi_1 \wedge \cdots \wedge d\pi_m$ with tuples $(f, \pi_1, \ldots, \pi_m)$ s.t. $f : Z \to \mathbb{R}$ is bounded Lipschitz and $\pi_i : Z \to \mathbb{R}$ are Lipschitz.

Given a Lipschitz $\varphi : M^m \to Z$, define a current acting on tuples:

$$\varphi \# [\mathcal{M}](f, \pi_1, \ldots, \pi_m) = \int_M (f \circ \varphi) \, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

Given a current $T$ acting on a tuple $\omega$ define $\partial T(\omega) = T(d\omega)$ where $d(f, \pi_1, \ldots, \pi_m) = (1, f, \pi_1, \ldots, \pi_m)$.

**Ambrosio-Kirchheim (2000):** an integral current, $T$, is an integer rectifiable current s.t. $\partial T$ is also integer rectifiable.

where an integer rectifiable current, $T$, has cntbly many pairwise disjoint biLip charts $\varphi_i : A_i \to \varphi_i(A_i) \subset Z$ and weights $a_i \in \mathbb{Z}$ s.t.

$$T(f, \pi_1, \ldots, \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi) \, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

of finite weighted volume: $\sum_{i=1}^{\infty} |a_i| \mathcal{H}^m(\varphi_i(A_i)) < \infty$. 
Integral Currents in Metric Spaces:

**DeGiorgi (1995):** For a complete metric space, $Z$, replace diff forms $f \, d\pi_1 \wedge \cdots \wedge d\pi_m$ with tuples $(f, \pi_1, \ldots, \pi_m)$ s.t. $f : Z \rightarrow \mathbb{R}$ is bounded Lipschitz and $\pi_i : Z \rightarrow \mathbb{R}$ are Lipschitz.

Given a Lipschitz $\varphi : M^m \rightarrow Z$, define a current acting on tuples:

$$\varphi \#[[M]](f, \pi_1, \ldots, \pi_m) = \int_M (f \circ \varphi) \, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

Given a current $T$ acting on a tuple $\omega$ define $\partial T(\omega) = T(d\omega)$ where $d(f, \pi_1, \ldots, \pi_m) = (1, f, \pi_1, \ldots, \pi_m)$.

**Ambrosio-Kirchheim (2000):** an integral current, $T$, is an integer rectifiable current s.t. $\partial T$ is also integer rectifiable. A current, $T$, has ctnbly many pairwise disjoint biLip charts $\varphi_i : A_i \rightarrow \varphi_i(A_i) \subset Z$ and weights $a_i \in \mathbb{Z}$ s.t.

$$T(f, \pi_1, \ldots, \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi) \, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

of finite weighted volume: $\sum_{i=1}^{\infty} |a_i| \mathcal{H}^m(\varphi_i(A_i)) < \infty$.

Mass is not the weighted volume in Ambrosio-Kirchheim Theory!
Mass of Integral Currents in Metric Spaces:

**Ambrosio-Kirchheim (2000):** an integral current, $T$, is an integer rectifiable current s.t. $\partial T$ is also integer rectifiable.

where an integer rectifiable current, $T$, has cntbly many pairwise disjoint biLip charts $\varphi_i : A_i \rightarrow \varphi_i(A_i) \subset Z$ and weights $a_i \in \mathbb{Z}$ s.t.

$$T(f, \pi_1, \ldots, \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

of finite weighted volume: $\sum_{i=1}^{\infty} |a_i| \mathcal{H}^m(\varphi_i(A_i)) < \infty$. 

**Defn [AK]:** The mass $M(T) = ||T||_Z$ where $||T||_T$ is the mass measure of $T$, which is the smallest measure $\mu_T$ s.t.

$$|\varphi # [M](fd\pi_1 \wedge \cdots \wedge d\pi_m)| \leq \prod_{i=1}^{m} \text{Lip}(\pi_i) \int \varphi(M) |f \circ \varphi| \mu_T$$

**Thm [AK]:** The mass measure $||T||_T = \lambda_\theta(H^m \text{set } T)$ where $\theta(p) = |a_i|$ if $p \in \varphi_i(A_i)$ and $\lambda(p) \in [c_m, C_m]$ and set $(T) = \{z \in Z \mid \lim \inf_{r \to 0} \frac{||T||_T(B(z, r))}{r^m} > 0\}$. 

Mass of Integral Currents in Metric Spaces:

**Ambrosio-Kirchheim (2000):** an integral current, $T$, is an integer rectifiable current s.t. $\partial T$ is also integer rectifiable.

where an integer rectifiable current, $T$, has countably many pairwise disjoint biLip charts $\varphi_i : A_i \to \varphi_i(A_i) \subset Z$ and weights $a_i \in \mathbb{Z}$ s.t.

$$T(f, \pi_1, \ldots, \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi) \, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

of finite weighted volume: $\sum_{i=1}^{\infty} |a_i| \mathcal{H}^m(\varphi_i(A_i)) < \infty$.

**Defn [AK]:** The mass $\mathbf{M}(T) = \|T\|(Z)$
Mass of Integral Currents in Metric Spaces:

**Ambrosio-Kirchheim (2000):** an integral current, \( T \), is an integer rectifiable current s.t. \( \partial T \) is also integer rectifiable.

where an integer rectifiable current, \( T \), has countably many pairwise disjoint biLip charts \( \varphi_i : A_i \to \varphi_i(A_i) \subset \mathbb{Z} \) and weights \( a_i \in \mathbb{Z} \) s.t.

\[
T(f, \pi_1, \ldots, \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi) \, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)
\]

of finite weighted volume: \( \sum_{i=1}^{\infty} |a_i| \mathcal{H}^m(\varphi_i(A_i)) < \infty \).

**Defn [AK]:** The mass \( \mathbf{M}(T) = \| T \|_\mathbb{Z} \)

where \( \| T \| = \mu_T \) is the mass measure of \( T \).
Mass of Integral Currents in Metric Spaces:

**Ambrosio-Kirchheim (2000):** an integral current, $T$, is an integer rectifiable current s.t. $\partial T$ is also integer rectifiable.

where an integer rectifiable current, $T$, has countably many pairwise disjoint biLip charts $\varphi_i : A_i \rightarrow \varphi_i(A_i) \subset Z$ and weights $a_i \in \mathbb{Z}$ s.t.

$$T(f, \pi_1, \ldots, \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi) \, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

of finite weighted volume: $\sum_{i=1}^{\infty} |a_i| \mathcal{H}^m(\varphi_i(A_i)) < \infty$.

**Defn [AK]:** The mass $M(T) = \|T\|(Z)$ where $\|T\| = \mu_T$ is the mass measure of $T$ which is the smallest measure $\mu$ s.t.

$$|\varphi_\#[[M]](f d \pi_1 \wedge \cdots \wedge d \pi_m) | \leq \prod_{i=1}^{m} \text{Lip}(\pi_i) \int_{\varphi(M)} |f \circ \varphi| \mu$$
Mass of Integral Currents in Metric Spaces:

Ambrosio-Kirchheim (2000): an integral current, $T$, is an integer rectifiable current s.t. $\partial T$ is also integer rectifiable. where an integer rectifiable current, $T$, has cntbly many pairwise disjoint biLip charts $\varphi_i : A_i \rightarrow \varphi_i(A_i) \subset \mathbb{Z}$ and weights $a_i \in \mathbb{Z}$ s.t.

$$T(f, \pi_1, \ldots, \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi) \, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

of finite weighted volume: $\sum_{i=1}^{\infty} |a_i| \mathcal{H}^m(\varphi_i(A_i)) < \infty$.

Defn [AK]: The mass $M(T) = \|T\|(\mathbb{Z})$

where $\|T\| = \mu_T$ is the mass measure of $T$

which is the smallest measure $\mu$ s.t.

$$|\varphi#[[M]](fd\pi_1 \wedge \cdots \wedge d\pi_m)| \leq \prod_{i=1}^{m} \text{Lip}(\pi_i) \int_{\varphi(M)} |f \circ \varphi| \mu$$

Thm [AK]: The mass measure $\|T\| = \lambda \theta (\mathcal{H}_m \mathbb{L} \text{ set } T)$
Mass of Integral Currents in Metric Spaces:

**Ambrosio-Kirchheim (2000):** an integral current, $T$, is an integer rectifiable current s.t. $\partial T$ is also integer rectifiable.

where an integer rectifiable current, $T$, has cntbly many pairwise disjoint biLip charts $\varphi_i : A_i \rightarrow \varphi_i(A_i) \subset \mathbb{Z}$ and weights $a_i \in \mathbb{Z}$ s.t.

\[
T(f, \pi_1, \ldots, \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)
\]

of finite weighted volume: $\sum_{i=1}^{\infty} |a_i| H^m(\varphi_i(A_i)) < \infty$.

**Defn [AK]:** The mass $M(T) = \|T\|(Z)$
where $\|T\| = \mu_T$ is the mass measure of $T$
which is the smallest measure $\mu$ s.t.

\[
|\varphi_#([M])(f d\pi_1 \wedge \cdots \wedge d\pi_m)| \leq \prod_{i=1}^{m} Lip(\pi_i) \int_{\varphi(M)} |f \circ \varphi| \mu
\]

**Thm [AK]:** The mass measure $\|T\| = \lambda \theta (H_m \text{ set } T)$
where $\theta(p) = |a_i|$ if $p \in \varphi_i(A_i)$
Mass of Integral Currents in Metric Spaces:

**Ambrosio-Kirchheim (2000):** an integral current, $T$, is an integer rectifiable current s.t. $\partial T$ is also integer rectifiable.

where an integer rectifiable current, $T$, has countably many pairwise disjoint biLip charts $\varphi_i : A_i \to \varphi_i(A_i) \subset Z$ and weights $a_i \in \mathbb{Z}$ s.t.

$$T(f, \pi_1, \ldots, \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

of finite weighted volume: $\sum_{i=1}^{\infty} |a_i| \mathcal{H}^m(\varphi_i(A_i)) < \infty$.

**Defn [AK]:** The mass $M(T) = \|T\|(Z)$ where $\|T\| = \mu_T$ is the mass measure of $T$ which is the smallest measure $\mu$ s.t.

$$|\varphi \# [[M]](fd\pi_1 \wedge \cdots \wedge d\pi_m)| \leq \prod_{i=1}^{m} \text{Lip}(\pi_i) \int_{\varphi(M)} |f \circ \varphi| \mu$$

**Thm [AK]:** The mass measure $\|T\| = \lambda \theta (\mathcal{H}_m \text{ set } T)$ where $\theta(p) = |a_i|$ if $p \in \varphi_i(A_i)$ and $\lambda(p) \in [c_m, C_m]$ and
Mass of Integral Currents in Metric Spaces:

**Ambrosio-Kirchheim (2000):** an integral current, $T$, is an integer rectifiable current s.t. $\partial T$ is also integer rectifiable.

where an integer rectifiable current, $T$, has countably many pairwise disjoint biLip charts $\varphi_i : A_i \to \varphi_i(A_i) \subset Z$ and weights $a_i \in \mathbb{Z}$ s.t.

$$T(f, \pi_1, \ldots, \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

of finite weighted volume: $\sum_{i=1}^{\infty} |a_i| \mathcal{H}^m(\varphi_i(A_i)) < \infty$.

**Defn [AK]:** The mass $M(T) = \|T\|(Z)$

where $\|T\| = \mu_T$ is the mass measure of $T$

which is the smallest measure $\mu$ s.t.

$$|\varphi_\#[[M]](fd\pi_1 \wedge \cdots \wedge d\pi_m)| \leq \prod_{i=1}^{m} \text{Lip}(\pi_i) \int_{\varphi(M)} |f \circ \varphi| \mu$$

**Thm [AK]:** The mass measure $\|T\| = \lambda \theta (\mathcal{H}_m \sqsubseteq \text{set } T)$

where $\theta(p) = |a_i|$ if $p \in \varphi_i(A_i)$ and $\lambda(p) \in [c_m, C_m]$ and

$$\text{set}(T) = \{z \in Z \mid \liminf_{r \to 0} \|T\|(B(z, r))/r^m > 0\}.$$
Ambrosio-Kirchheim Compactness Theorem:

Ambrosio-Kirchheim (2000): an integral current, $T$, is an integer rectifiable current s.t. $\partial T$ is also integer rectifiable.

where an integer rectifiable current, $T$, has cntbly many pairwise disjoint biLip charts $\varphi_i : A_i \to \varphi_i(A_i) \subset Z$ and weights $a_i \in \mathbb{Z}$ s.t.

$$T(f, \pi_1, ..., \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

They define mass $M(T) = \|T\|(Z)$ where $\|T\| = \lambda \theta (H_m \restriction \text{set} T)$ where $\theta(p) = a_i$ if $p \in \varphi_i(A_i)$ and $\lambda(p) \in [c_m, C_m]$ and

$$\text{set}(T) = \{z \in Z | \liminf_{r \to 0} \|T\|(B(z, r))/r^m > 0\}.$$
Ambrosio-Kirchheim Compactness Theorem:

**Ambrosio-Kirchheim (2000):** an integral current, $T$, is an integer rectifiable current s.t. $\partial T$ is also integer rectifiable.

where an integer rectifiable current, $T$, has cntbly many pairwise disjoint biLip charts $\varphi_i : A_i \to \varphi_i(A_i) \subset Z$ and weights $a_i \in \mathbb{Z}$ s.t.

$$T(f, \pi_1, \ldots, \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi) \, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

They define mass $M(T) = \|T\|(Z)$ where $\|T\| = \lambda \theta (\mathcal{H}_m \sqcap \text{set } T)$

where $\theta(p) = a_i$ if $p \in \varphi_i(A_i)$ and $\lambda(p) \in [c_m, C_m]$ and

$$\text{set}(T) = \{z \in Z \mid \liminf_{r \to 0} \|T\|(B(z, r))/r^m > 0\}.$$  

Note: $\text{set}(T)$ is cntbly rectifiable: $\mathcal{H}^m (\text{set}(T) \setminus \bigcup_{i=1}^{\infty} \varphi_i(A_i)) = 0$. 


Ambrosio-Kirchheim Compactness Theorem:

**Ambrosio-Kirchheim (2000):** an integral current, $T$, is an integer rectifiable current s.t. $\partial T$ is also integer rectifiable.

where an integer rectifiable current, $T$, has cntbly many pairwise disjoint biLip charts $\varphi_i : A_i \to \varphi_i(A_i) \subset Z$ and weights $a_i \in \mathbb{Z}$ s.t.

$$T(f, \pi_1, \ldots, \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

They define mass $M(T) = \| T \|(Z)$ where $\| T \| = \lambda \theta (\mathcal{H}_m \upharpoonright \text{set } T)$

where $\theta(p) = a_i$ if $p \in \varphi_i(A_i)$ and $\lambda(p) \in [c_m, C_m]$ and

$$\text{set}(T) = \{ z \in Z \mid \liminf_{r \to 0} \| T \|(B(z, r))/r^m > 0 \}.$$ 

Note: $\text{set}(T)$ is cntbly rectifiable: $\mathcal{H}^m (\text{set}(T) \setminus \bigcup_{i=1}^{\infty} \varphi_i(A_i)) = 0.$

**Compactness Thm [AK]:** If integral currents $T_j$ have

$\text{set}(T_j) \subset K$ compact, and $M(T_j) \leq V$ and $M(\partial T_j) \leq A$ then

$\exists$ subseq $T_{j_k}$ and an integral current $T_\infty$ s.t. $T_j(\omega) \to T_\infty(\omega) \forall \omega$
Ambrosio-Kirchheim Compactness Theorem:

**Ambrosio-Kirchheim (2000):** an integral current, $T$, is an integer rectifiable current s.t. $\partial T$ is also integer rectifiable. 

where an integer rectifiable current, $T$, has cntbly many pairwise disjoint biLip charts $\varphi_i : A_i \to \varphi_i(A_i) \subset \mathbb{Z}$ and weights $a_i \in \mathbb{Z}$ s.t.

$$T(f, \pi_1, ..., \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

They define mass $M(T) = \| T \|(\mathbb{Z})$ where $\| T \| = \lambda \theta (\mathcal{H}_m \sqcap \text{set } T)$

where $\theta(p) = a_i$ if $p \in \varphi_i(A_i)$ and $\lambda(p) \in [c_m, C_m]$ and

$$\text{set}(T) = \{ z \in \mathbb{Z} \mid \liminf_{r \to 0} \| T \|(B(z, r))/r^m > 0 \}.$$ 

Note: $\text{set}(T)$ is cntbly rectifiable: $\mathcal{H}^m(\text{set}(T) \setminus \bigcup_{i=1}^{\infty} \varphi_i(A_i)) = 0$.

**Compactness Thm [AK]:** If integral currents $T_j$ have $\text{set}(T_j) \subset K$ compact, and $M(T_j) \leq V$ and $M(\partial T_j) \leq A$ then $\exists$ subseq $T_{j_k}$ and an integral current $T_\infty$ s.t. $T_j(\omega) \to T_\infty(\omega) \forall \omega$ and $\partial T_j(\eta) \to \partial T_\infty(\eta) \forall \eta$
**Ambrosio-Kirchheim Compactness Theorem:**

**Ambrosio-Kirchheim (2000):** an integral current, \( T \), is an integer rectifiable current s.t. \( \partial T \) is also integer rectifiable. where an integer rectifiable current, \( T \), has cntbly many pairwise disjoint biLip charts \( \varphi_i : A_i \to \varphi_i(A_i) \subset Z \) and weights \( a_i \in \mathbb{Z} \) s.t.

\[
T(f, \pi_1, ..., \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)
\]

They define mass \( M(T) = \| T \|(Z) \) where \( \| T \| = \lambda \theta (\mathcal{H}_m \ll \text{set } T) \)

where \( \theta(p) = a_i \) if \( p \in \varphi_i(A_i) \) and \( \lambda(p) \in [c_m, C_m] \) and

\[
\text{set}(T) = \{ z \in Z | \lim_{r \to 0} \| T \|(B(z, r))/r^m > 0 \}.
\]

Note: \( \text{set}(T) \) is cntbly rectifiable: \( \mathcal{H}^m(\text{set}(T) \setminus \bigcup_{i=1}^{\infty} \varphi_i(A_i)) = 0. \)

**Compactness Thm [AK]:** If integral currents \( T_j \) have \( \text{set}(T_j) \subset K \) compact, and \( M(T_j) \leq V \) and \( M(\partial T_j) \leq A \) then \( \exists \) subseq \( T_{j_k} \) and an integral current \( T_\infty \) s.t. \( T_j(\omega) \to T_\infty(\omega) \) \( \forall \omega \) and \( \partial T_j(\eta) \to \partial T_\infty(\eta) \) \( \forall \eta \) and \( \lim inf_{j \to \infty} M(T_j) \geq M(T_\infty). \)
Sormani-Wenger: Intrinsic Flat Distance

The *intrinsic flat distance* between oriented manifolds $M_i^m$ is:

$$d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d_Z^F (\varphi_1#[[M_1^m]], \varphi_2#[[M_2^m]]) \mid \varphi_i : M_i^m \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i^m \to Z$.

$d_Z^F (\varphi_1#[[M_1^m]], \varphi_2#[[M_2^m]])$ is the inf over integral currents $A, B$

$$= \inf \left\{ \mathbf{M}_{area}(A) + \mathbf{M}_{vol}(B) : A + \partial B = \varphi_1#[[M_1^m]] - \varphi_2#[[M_2^m]] \right\}$$

Thm [SW-JDG]: If $M_i^m$ are compact and $d_{SWIF}(M_1^m, M_2^m) = 0$ then there exists an orientation preserving isometry $F : M_1^m \to M_2^m$.

Pf: There exists $\varphi_i : M_i^m \to Z$ such that $\varphi_1#[[M_1^m]] = \varphi_2#[[M_2^m]]$.

Let $F = \varphi_2^{-1} \circ \varphi_1$.

Next: We need to define the SWIF limit spaces!
Sormani-Wenger: Intrinsic Flat Distance

The *intrinsic flat distance* between oriented manifolds $M_i^m$ is:

$$d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d_Z^F (\varphi_1#[[M_1^m]], \varphi_2#[[M_2^m]]) \mid \varphi_i : M_i^m \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i^m \to Z$.

$d_Z^F (\varphi_1#[[M_1^m]], \varphi_2#[[M_2^m]])$ is the inf over integral currents $A, B$

$$= \inf \left\{ \mathbf{M}_{area}(A) + \mathbf{M}_{vol}(B) : A + \partial B = \varphi_1#[[M_1^m]] - \varphi_2#[[M_2^m]] \right\}$$

Recall: integral currents act on tuples of Lip fnctns $(f, \pi_1, ..., \pi_m)$.
Sormani-Wenger: Intrinsic Flat Distance

The *intrinsic flat distance* between oriented manifolds $M_i^m$ is:

$$d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d_F^Z (\varphi_1 #[[M_1^m]], \varphi_2 #[[M_2^m]]) \mid \varphi_i : M_i^m \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i^m \to Z$.

$$d_F^Z (\varphi_1 #[[M_1^m]], \varphi_2 #[[M_2^m]])$$ is the inf over integral currents $A$ and $B$

$$= \inf \left\{ \text{area}(A) + \text{vol}(B) : A + \partial B = \varphi_1 #[[M_1^m]] - \varphi_2 #[[M_2^m]] \right\}$$

Recall: integral currents act on tuples of Lip functions $(f, \pi_1, \ldots, \pi_m)$

$$\varphi #[[M]](f, \pi_1, \ldots, \pi_m) = \int_M (f \circ \varphi) \, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$
Sormani-Wenger: Intrinsic Flat Distance

The *intrinsic flat distance* between oriented manifolds $M_i^m$ is:

$$d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d_F^Z (\varphi_1\#[[M_1^m]], \varphi_2\#[[M_2^m]]) \mid \varphi_i : M_i^m \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i^m \to Z$.

$d_F^Z (\varphi_1\#[[M_1^m]], \varphi_2\#[[M_2^m]])$ is the inf over integral currents $\begin{array}{l} A \quad B \end{array}$

$$= \inf \left\{ M_{\text{area}}(A) + M_{\text{vol}}(B) : A + \partial B = \varphi_1\#[[M_1^m]] - \varphi_2\#[[M_2^m]] \right\}$$

Recall: integral currents act on tuples of Lip fnctns $(f, \pi_1, ..., \pi_m)$

$$\varphi\#[[M]](f, \pi_1, ..., \pi_m) = \int_M (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

and $\partial B(\omega) = B(d\omega)$ where $d(f, \pi_1, ..., \pi_m) = (1, f, \pi_1, ..., \pi_m)$. 

Thm [SW-JDG]: If $M_i$ are compact and $d_{SWIF}(M_1^m, M_2^m) = 0$ then $\exists$ orientation preserving isometry $F : M_1^m \to M_2^m$.

Pf: $\exists \varphi_i : M_i^m \to Z$ s.t. $\varphi_1\#[[M_1^m]] = \varphi_2\#[[M_2^m]]$.

Let $F = \varphi_2^{-1} \circ \varphi_1$.

Next: We need to define the SWIF limit spaces!
Sormani-Wenger: Intrinsic Flat Distance

The *intrinsic flat distance* between oriented manifolds $M_i^m$ is:

$$d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d_Z^F (\varphi_1\#[[M_1^m]], \varphi_2\#[[M_2^m]]) \mid \varphi_i : M_i^m \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i^m \to Z$.

$d_Z^F (\varphi_1\#[[M_1^m]], \varphi_2\#[[M_2^m]])$ is the inf over integral currents $A \wedge B$

$$= \inf \left\{ \mathbf{M}_{area}(A) + \mathbf{M}_{vol}(B) : A + \partial B = \varphi_1\#[[M_1^m]] - \varphi_2\#[[M_2^m]] \right\}$$

Recall: integral currents act on tuples of Lip fnctns $(f, \pi_1, ..., \pi_m)$

$\varphi\#[[M]](f, \pi_1, ..., \pi_m) = \int_M (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$

and $\partial B(\omega) = B(d\omega)$ where $d(f, \pi_1, ..., \pi_m) = (1, f, \pi_1, ..., \pi_m)$. 
Sormani-Wenger: Intrinsic Flat Distance

The *intrinsic flat distance* between oriented manifolds $M^m_i$ is:

$$d_{SWIF}(M^m_1, M^m_2) = \inf \left\{ d^Z_F (\varphi_1\#[[M^m_1]], \varphi_2\#[[M^m_2]]) \mid \varphi_i : M^m_i \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M^m_i \to Z$.

$d^Z_F (\varphi_1\#[[M^m_1]], \varphi_2\#[[M^m_2]])$ is the inf over integral currents $\text{A} - \text{B}$

$$= \inf \left\{ M_{\text{area}}(\text{A}) + M_{\text{vol}}(\text{B}) : \text{A} + \partial \text{B} = \varphi_1\#[[M^m_1]] - \varphi_2\#[[M^m_2]] \right\}$$

Recall: integral currents act on tuples of Lip fnctns $(f, \pi_1, ..., \pi_m)$

$$\varphi\#[[M]](f, \pi_1, ..., \pi_m) = \int_M (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge ... \wedge d(\pi_m \circ \varphi)$$

and $\partial B(\omega) = B(d\omega)$ where $d(f, \pi_1, ..., \pi_m) = (1, f, \pi_1, ..., \pi_m)$.

**Thm [SW-JDG]:** If $M_i$ are compact and $d_{SWIF}(M_1, M_2) = 0$ then $\exists$ orientation preserving isometry $F : M_1 \to M_2$. 
Sormani-Wenger: Intrinsic Flat Distance

The *intrinsic flat distance* between oriented manifolds $M^m_i$ is:

$$d_{SWIF}(M^m_1, M^m_2) = \inf \left\{ d_Z^F(\varphi_1#[[M^m_1]], \varphi_2#[[M^m_2]]) \mid \varphi_i : M^m_i \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M^m_i \to Z$.

$d_Z^F(\varphi_1#[[M^m_1]], \varphi_2#[[M^m_2]])$ is the inf over integral currents

$$= \inf \left\{ \begin{array}{l} \text{M}_{area}(A) + \text{M}_{vol}(B) : A + \partial B = \varphi_1#[[M^m_1]] - \varphi_2#[[M^m_2]] \end{array} \right\}$$

Recall: integral currents act on tuples of Lip fnctns $(f, \pi_1, ..., \pi_m)$

$$\varphi#[[M]](f, \pi_1, ..., \pi_m) = \int_M (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

and $\partial B(\omega) = B(d\omega)$ where $d(f, \pi_1, ..., \pi_m) = (1, f, \pi_1, ..., \pi_m)$.

**Thm [SW-JDG]:** If $M_i$ are compact and $d_{SWIF}(M_1, M_2) = 0$ then $\exists$ orientation preserving isometry $F : M_1 \to M_2$.

**Pf:**
Sormani-Wenger: Intrinsic Flat Distance

The *intrinsic flat distance* between oriented manifolds $M_i^m$ is:

$$d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d^Z_F (\varphi_1#, [[M_1^m]], \varphi_2#([[M_2^m]])) \mid \varphi_i : M_i^m \rightarrow Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i^m \rightarrow Z$.

$d^Z_F (\varphi_1#, [[M_1^m]], \varphi_2#([[M_2^m]]))$ is the inf over integral currents

$$= \inf \left\{ M_{\text{area}}(A) + M_{\text{vol}}(B) : A + \partial B = \varphi_1#([[M_1^m]]) - \varphi_2#([ [M_2^m]]) \right\}$$

Recall: integral currents act on tuples of Lip fnctns $(f, \pi_1, ..., \pi_m)$

$$\varphi#[[M]](f, \pi_1, ..., \pi_m) = \int_M (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge ... \wedge d(\pi_m \circ \varphi)$$

and $\partial B(\omega) = B(d\omega)$ where $d(f, \pi_1, ..., \pi_m) = (1, f, \pi_1, ..., \pi_m)$.

**Thm [SW-JDG]:** If $M_i$ are compact and $d_{SWIF}(M_1, M_2) = 0$ then $\exists$ orientation preserving isometry $F : M_1 \rightarrow M_2$.

**Pf:** $\exists \varphi_i : M_i \rightarrow Z$ s.t. $\varphi_1#[[M_1]] = \varphi_2#[[M_2]]$. 
Sormani-Wenger: Intrinsic Flat Distance

The intrinsic flat distance between oriented manifolds $M^m_i$ is:

$$d_{SWIF}(M^m_1, M^m_2) = \inf \left\{ d^Z_F (\varphi_1#[[M^m_1]], \varphi_2#[[M^m_2]]) \mid \varphi_i : M^m_i \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M^m_i \to Z$.

$$d^Z_F (\varphi_1#[[M^m_1]], \varphi_2#[[M^m_2]])$$ is the inf over integral currents $A, B$

$$= \inf \left\{ \text{M}_{\text{area}}(A) + \text{M}_{\text{vol}}(B) : A + \partial B = \varphi_1#[[M^m_1]] - \varphi_2#[[M^m_2]] \right\}$$

Recall: integral currents act on tuples of Lip fnctns $(f, \pi_1, \ldots, \pi_m)$

$\varphi#[[M]](f, \pi_1, \ldots, \pi_m) = \int_M (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$

and $\partial B(\omega) = B(d\omega)$ where $d(f, \pi_1, \ldots, \pi_m) = (1, f, \pi_1, \ldots, \pi_m)$.

**Thm [SW-JDG]:** If $M_i$ are compact and $d_{SWIF}(M_1, M_2) = 0$ then $\exists$ orientation preserving isometry $F : M_1 \to M_2$.

**Pf:** $\exists \varphi_i : M_i \to Z$ s.t. $\varphi_1#[[M_1]] = \varphi_2#[[M_2]]$. Let $F = \varphi_2^{-1} \circ \varphi_1$. 

Sormani-Wenger: Intrinsic Flat Distance

The intrinsic flat distance between oriented manifolds $M_i^m$ is:

$$d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d^Z_F (\varphi_1#[[M_1^m]], \varphi_2#[[M_2^m]]) \mid \varphi_i : M_i^m \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i^m \to Z$.

$d^Z_F (\varphi_1#[[M_1^m]], \varphi_2#[[M_2^m]])$ is the inf over integral currents $A, B$

$$= \inf \left\{ M \left( \begin{array}{c} A \end{array} \right) + M \left( \begin{array}{c} B \end{array} \right) : A + \partial B = \varphi_1#[[M_1^m]] - \varphi_2#[[M_2^m]] \right\}$$

Recall: integral currents act on tuples of Lip fnctns $(f, \pi_1, ..., \pi_m)$

$$\varphi#[[M]](f, \pi_1, ..., \pi_m) = \int_M (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

and $\partial B(\omega) = B(d\omega)$ where $d(f, \pi_1, ..., \pi_m) = (1, f, \pi_1, ..., \pi_m)$.

**Thm [SW-JDG]:** If $M_i$ are compact and $d_{SWIF}(M_1, M_2) = 0$

then $\exists$ orientation preserving isometry $F : M_1 \to M_2$.

**Pf:** $\exists \varphi_i : M_i \to Z$ s.t. $\varphi_1#[[M_1]] = \varphi_2#[[M_2]]$. Let $F = \varphi_2^{-1} \circ \varphi_1$. 
Sormani-Wenger: Intrinsic Flat Distance

The *intrinsic flat distance* between oriented manifolds $M_i^m$ is:

$$d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d_Z^F (\varphi_1#[[M_1^m]], \varphi_2#[[M_2^m]]) \mid \varphi_i : M_i^m \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i^m \to Z$.

$d_Z^F (\varphi_1#[[M_1^m]], \varphi_2#[[M_2^m]])$ is the inf over integral currents $A$, $B$

$$= \inf \left\{ M_{\text{area}}(A) + M_{\text{vol}}(B) : A + \partial B = \varphi_1#[[M_1^m]] - \varphi_2#[[M_2^m]] \right\}$$

Recall: integral currents act on tuples of Lip fnctns $(f, \pi_1, \ldots, \pi_m)$

$$\varphi#[[M]](f, \pi_1, \ldots, \pi_m) = \int_M (f \circ \varphi) \, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

and $\partial B(\omega) = B(d\omega)$ where $d(f, \pi_1, \ldots, \pi_m) = (1, f, \pi_1, \ldots, \pi_m)$.

**Thm [SW-JDG]:** If $M_i$ are compact and $d_{SWIF}(M_1, M_2) = 0$ then $\exists$ orientation preserving isometry $F : M_1 \to M_2$.

**Pf:** $\exists \varphi_i : M_i \to Z$ s.t. $\varphi_1#[[M_1]] = \varphi_2#[[M_2]]$. Let $F = \varphi_2^{-1} \circ \varphi_1$.

Next: We need to define the SWIF limit spaces!
Sormani-Wenger: Intrinsic Flat Distance

The intrinsic flat distance between oriented manifolds $M_i^m$ is:

$$d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d^Z_F (\varphi_1#[[M_1^m]], \varphi_2#[[M_2^m]]) \mid \varphi_i : M_i^m \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i^m \to Z$.

$$d^Z_F (\varphi_1#[[M_1^m]], \varphi_2#[[M_2^m]])$$ is the inf over integral currents $\text{A} \bigcup \text{B}$

$$= \inf \left\{ \begin{array}{c} \text{M}_{\text{area}}(\text{A}) + \text{M}_{\text{vol}}(\text{B}) : \text{A} + \partial \text{B} = \varphi_1#[[M_1^m]] - \varphi_2#[[M_2^m]] \end{array} \right\}$$

Recall: integral currents act on tuples of Lip fnctns $(f, \pi_1, \ldots, \pi_m)$

$$\varphi#[[M]](f, \pi_1, \ldots, \pi_m) = \int_M (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

and $\partial B(\omega) = B(d\omega)$ where $d(f, \pi_1, \ldots, \pi_m) = (1, f, \pi_1, \ldots, \pi_m)$.

**Thm [SW-JDG]:** If $M_i$ are compact and $d_{SWIF}(M_1, M_2) = 0$ then $\exists$ orientation preserving isometry $F : M_1 \to M_2$.

**Pf:** $\exists \varphi_i : M_i \to Z$ s.t. $\varphi_1#[[M_1]] = \varphi_2#[[M_2]]$. Let $F = \varphi_2^{-1} \circ \varphi_1$.

Next: We need to define the SWIF limit spaces!
Sormani-Wenger: Intrinsic Flat Distance

The intrinsic flat distance between oriented manifolds \( M_i^m \) is:

\[
d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d_Z^F(\varphi_1#[[M_1^m]], \varphi_2#[[M_2^m]]) \mid \varphi_i : M_i^m \to Z \right\}
\]

where the infimum is taken over all complete metric spaces, \( Z \), and over all distance preserving maps \( \varphi_i : M_i^m \to Z \).

\[
d_Z^F(\varphi_1#[[M_1^m]], \varphi_2#[[M_2^m]]) \text{ is the inf over integral currents } A, B \]

\[
= \inf \left\{ \begin{array}{c}
\text{area}(A) + \text{vol}(B) : A + \partial B = \varphi_1#[[M_1^m]] - \varphi_2#[[M_2^m]]
\end{array} \right\}
\]

Recall: integral currents act on tuples of Lip fnctns \((f, \pi_1, \ldots, \pi_m)\)

\[
\varphi_#[[M]](f, \pi_1, \ldots, \pi_m) = \int_M (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)
\]

and \( \partial B(\omega) = B(d\omega) \) where \( d(f, \pi_1, \ldots, \pi_m) = (1, f, \pi_1, \ldots, \pi_m) \).

**Thm [SW-JDG]:** If \( M_i \) are compact and \( d_{SWIF}(M_1, M_2) = 0 \) then \( \exists \) orientation preserving isometry \( F : M_1 \to M_2 \).

**Pf:** \( \exists \varphi_i : M_i \to Z \) s.t. \( \varphi_1#[[M_1]] = \varphi_2#[[M_2]] \). Let \( F = \varphi_2^{-1} \circ \varphi_1 \).

Next: We need to define the SWIF limit spaces!
SWIF Limits: Integral Current Spaces

A sequence of compact Riemannian manifolds can converge in the intrinsic flat (SWIF) sense to the following limit which is an integral current space:
SWIF Limits: Integral Current Spaces

A sequence of compact Riemannian manifolds can converge in the intrinsic flat (SWIF) sense to the following limit which is an integral current space:

Vague Definition from Lesson 1:

Defn: An Integral Current Space is m-rectifiable (which means it has countably many bi-Lipschitz charts of the same dimension m as the original sequence) and it has a well defined (m-1)-rectifiable boundary. The charts are oriented and have integer valued weights, $\theta$.
SWIF Limits: Integral Current Spaces

A sequence of compact Riemannian manifolds can converge in the intrinsic flat (SWIF) sense to the following limit which is an integral current space:

Vague Definition from Lesson 1:

Defn: An Integral Current Space is m-rectifiable (which means it has countably many bi-Lipschitz charts of the same dimension m as the original sequence) and it has a well defined (m-1)-rectifiable boundary. The charts are oriented and have integer valued weights, $\theta$. 
SWIF Limits: Integral Current Spaces

A sequence of compact Riemannian manifolds can converge in the intrinsic flat (SWIF) sense to the following limit which is an integral current space:

[Diagram showing a sequence of geometric structures converging to a limit]

Vague Definition from Lesson 1:

Defn: An Integral Current Space is m-rectifiable (which means it has countably many bi-Lipschitz charts of the same dimension m as the original sequence) and it has a well defined (m-1)-rectifiable boundary. The charts are oriented and have integer valued weights, $\theta$.

Now we can truly define integral current spaces.
SWIF limits are Integral Current Spaces

Recall Flat limits of oriented submanifolds are integral currents:

Ambrosio-Kirchheim (2000):
an integral current, $T$, on $Z$ is an integer rectifiable current s.t.

$\partial T$ is also integer rectifiable.

Defn: an integer rectifiable current, $T$, has cntbly many pairwise

disjoint biLip charts $\phi_i: A_i \to \phi_i(A_i) \subset Z$ and weights $a_i \in \mathbb{Z}$ s.t.

$T(f, \pi_1, \ldots, \pi_m) = \infty \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \phi) d(\pi_1 \circ \phi) \wedge \ldots \wedge d(\pi_m \circ \phi)$

with mass $M(T) = ||T||(Z)$ where

$||T|| = \lambda_{\theta}(H^m set T)$ and $set(T) = \{ z \in Z | \lim inf_{r \to 0} ||T||(B(z, r))/r^m > 0 \}$.

Key Idea: Integral currents generalize oriented submanifolds in $Z$.

Key New Idea: generalize oriented Riemannian Manifolds.

Sormani-Wenger Defn: An integral current space $M = (X, d, T)$ is a metric space $(X, d)$ and integral current $T$ s.t.

$set(T) = X$.

Furthermore:

$M(M) = M(T)$ and $\partial M = (set(\partial T), d, \partial T)$.

Thus $X$ is cntbly $H^m$ rectifiable: it has cntbly many pairwise

disjoint Lip charts $\phi_i: A_i \to X$ s.t.

$H^m(X \setminus \bigcup_{i=1}^{\infty} \phi_i(A_i)) = 0$. 
**SWIF limits are Integral Current Spaces**

Recall Flat limits of **oriented submanifolds** are **integral currents**:  

**Ambrosio-Kirchheim (2000):** an integral current, $T$, on $Z$ is an integer rectifiable current s.t. $\partial T$ is also integer rectifiable.  

**Defn:** an integer rectifiable current, $T$, has cntbly many pairwise disjoint biLip charts $\varphi_i : A_i \to \varphi_i(A_i) \subset Z$ and weights $a_i \in \mathbb{Z}$ s.t.

$$T(f, \pi_1, ..., \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

with **mass** $M(T) = \|T\|(Z)$ where $\|T\| = \lambda \theta (\mathcal{H}_m \perp \text{set } T)$ and  

$$\text{set}(T) = \{ z \in Z | \liminf_{r \to 0} \|T\|(B(z, r))/r^m > 0 \}.$$
SWIF limits are Integral Current Spaces

Recall Flat limits of oriented submanifolds are integral currents: **Ambrosio-Kirchheim (2000):** an integral current, \( T \), on \( Z \) is an integer rectifiable current s.t. \( \partial T \) is also integer rectifiable.

**Defn:** an integer rectifiable current, \( T \), has cntbly many pairwise disjoint biLip charts \( \varphi_i : A_i \to \varphi_i(A_i) \subset Z \) and weights \( a_i \in \mathbb{Z} \) s.t.

\[
T(f, \pi_1, \ldots, \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)
\]

with mass \( M(T) = \| T \| (Z) \) where \( \| T \| = \lambda \theta (\mathcal{H}_m \downarrow \text{set } T) \) and

\[
\text{set}(T) = \{ z \in Z \mid \liminf_{r \to 0} \| T \|(B(z, r))/r^m > 0 \}.
\]

**Key Idea:** Integral currents generalize oriented submanifolds in \( Z \).
SWIF limits are Integral Current Spaces

Recall Flat limits of **oriented submanifolds** are **integral currents**: 

**Ambrosio-Kirchheim (2000):** an integral current, \( T \), on \( Z \) is an integer rectifiable current s.t. \( \partial T \) is also integer rectifiable.

**Defn:** an **integer rectifiable current**, \( T \), has cntbly many pairwise disjoint biLip charts \( \varphi_i : A_i \to \varphi_i(A_i) \subset Z \) and weights \( a_i \in \mathbb{Z} \) s.t.

\[
T(f, \pi_1, \ldots, \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)
\]

with **mass** \( \mathbf{M}(T) = \| T \| (Z) \) where \( \| T \| = \lambda \theta (\mathcal{H}_m \downarrow \text{set} T) \) and

\[
\text{set}(T) = \{ z \in Z | \liminf_{r \to 0} \| T \|(B(z, r))/r^m > 0 \}.
\]

**Key Idea:** Integral currents generalize **oriented submanifolds** in \( Z \).

**Key New Idea:** generalize **oriented Riemannian Manifolds**.
SWIF limits are Integral Current Spaces

Recall Flat limits of oriented submanifolds are integral currents: **Ambrosio-Kirchheim (2000):** an integral current, $T$, on $Z$ is an integer rectifiable current s.t. $\partial T$ is also integer rectifiable.

**Defn:** an integer rectifiable current, $T$, has cntbly many pairwise disjoint biLip charts $\varphi_i : A_i \to \varphi_i(A_i) \subset Z$ and weights $a_i \in \mathbb{Z}$ s.t.

$$T(f, \pi_1, ..., \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi) \, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

with mass $M(T) = \|T\|(Z)$ where $\|T\| = \lambda \theta(H_m \perp \text{set } T)$ and

$$\text{set}(T) = \{ z \in Z \mid \liminf_{r \to 0} \|T\|(B(z, r)) / r^m > 0 \}.$$ 

**Key Idea:** Integral currents generalize oriented submanifolds in $Z$.

**Key New Idea:** generalize oriented Riemannian Manifolds.

**Sormani-Wenger Defn:** An integral current space $M = (X, d, T)$ is a metric space $(X, d)$ and integral current $T$ s.t. $\text{set}(T) = X$. 
SWIF limits are Integral Current Spaces

Recall Flat limits of oriented submanifolds are integral currents: Ambrosio-Kirchheim (2000): an integral current, $T$, on $Z$ is an integer rectifiable current s.t. $\partial T$ is also integer rectifiable.

**Defn:** an integer rectifiable current, $T$, has cntbly many pairwise disjoint biLip charts $\varphi_i : A_i \to \varphi_i(A_i) \subset Z$ and weights $a_i \in \mathbb{Z}$ s.t.

$$T(f, \pi_1, ..., \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

with mass $M(T) = \|T\|(Z)$ where $\|T\| = \lambda \theta (H_m \perp \text{set } T)$ and

$$\text{set}(T) = \{ z \in Z | \liminf_{r \to 0} \|T\|(B(z, r))/r^m > 0 \}.$$

**Key Idea:** Integral currents generalize oriented submanifolds in $Z$.

**Key New Idea:** generalize oriented Riemannian Manifolds.

**Sormani-Wenger Defn:** An integral current space $M = (X, d, T)$ is a metric space $(X, d)$ and integral current $T$ s.t. $\text{set}(T) = X$.

Furthermore: $M(M) = M(T)$ and $\partial M = (\text{set}(\partial T), d, \partial T)$. 

SWIF limits are Integral Current Spaces

Recall Flat limits of oriented submanifolds are integral currents: 
**Ambrosio-Kirchheim (2000):** an integral current, $T$, on $Z$ is an integer rectifiable current s.t. $\partial T$ is also integer rectifiable.

**Defn:** an integer rectifiable current, $T$, has cntbly many pairwise disjoint biLip charts $\varphi_i : A_i \to \varphi_i(A_i) \subset Z$ and weights $a_i \in \mathbb{Z}$ s.t.

$$T(f, \pi_1, \ldots, \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

with mass $M(T) = ||T||(Z)$ where $||T|| = \lambda \theta (\mathcal{H}_m \sqsubset \text{set } T)$ and

$$\text{set}(T) = \{ z \in Z \mid \liminf_{r \to 0} ||T||(B(z, r))/r^m > 0 \}.$$

**Key Idea:** Integral currents generalize oriented submanifolds in $Z$.

**Key New Idea:** generalize oriented Riemannian Manifolds.

**Sormani-Wenger Defn:** An integral current space $M = (X, d, T)$ is a metric space $(X, d)$ and integral current $T$ s.t. $\text{set}(T) = X$. Furthermore: $M(M) = M(T)$ and $\partial M = (\text{set}(\partial T), d, \partial T)$.

Thus $X$ is cntbly $\mathcal{H}_m$ rectifiable: it has cntbly many pairwise disjoint Lip charts $\varphi_i : A_i \to X$ s.t. $\mathcal{H}_m (X \setminus \bigcup_{i=1}^{\infty} \varphi_i(A_i)) = 0.$
Oriented Riemannian Mnflds and Integral Current Spaces
An oriented Riemannian mnfld $\left( M^m, g \right)$ is a metric space $\left( M, d_M \right)$
Oriented Riemannian Manifolds and Integral Current Spaces

An oriented Riemannian manifold \((M^m, g)\) is a metric space \((M, d_M)\) with a smooth collection of charts

\[
d_M(p, q) = \inf \{ L_g(C) : C : [0, 1] \to M, \ C(0) = p, \ C(1) = q \}
\]

where \(L_g(C) = \int_0^1 g(C'(s), C'(s))^{1/2} \, ds\)

\[
Vol(U) = \mathcal{H}^m(U) \quad \text{is the Hausdorff measure.}
\]
Oriented Riemannian Manifolds and Integral Current Spaces

An oriented Riemannian manifold \((M^m, g)\) is a metric space \((M, d_M)\) with a smooth collection of charts

\[
d_M(p, q) = \inf \{ L_g(C) : C : [0, 1] \to M, \; C(0) = p, \; C(1) = q \}
\]

where \(L_g(C) = \int_0^1 g(C'(s), C'(s))^{1/2} \, ds\)

\(Vol(U) = \mathcal{H}^m(U)\) is the Hausdorff measure.

An integral current space \(M = (X, d, T)\) is a metric space \((X, d)\) and integral current \(T\) s.t. \(\text{set}(T) = X\) where

\[
\text{set}(T) = \{ z \in Z | \liminf_{r \to 0} ||T||(B(z, r))/r^m > 0 \}
\]

So it has a countable collection of biLipschitz charts.
Oriented Riemannian Manifolds and Integral Current Spaces

An oriented Riemannian manifold $(M^m, g)$ is a metric space $(M, d_M)$ with a smooth collection of charts

$$d_M(p, q) = \inf \{ L_g(C) : C : [0, 1] \to M, \ C(0) = p, \ C(1) = q \}$$

where $L_g(C) = \int_0^1 g(C'(s), C'(s))^{1/2} \, ds$

$Vol(U) = \mathcal{H}^m(U)$ is the Hausdorff measure.

An integral current space $M = (X, d, T)$ is a metric space $(X, d)$ and integral current $T$ s.t. $\text{set}(T) = X$ where

$\text{set}(T) = \{ z \in Z | \liminf_{r \to 0} ||T||(B(z, r))/r^m > 0 \}$

So it has a countable collection of biLipschitz charts that are oriented and weighted by $\theta : M \to \mathbb{Z}$
An oriented Riemannian manifold \((M^m, g)\) is a metric space \((M, d_M)\) with a smooth collection of charts

\[
d_M(p, q) = \inf \{L_g(C) : C : [0, 1] \to M, \ C(0) = p, \ C(1) = q\}
\]

where

\[
L_g(C) = \int_0^1 g(C'(s), C'(s))^{1/2} \, ds
\]

\[
\text{Vol}(U) = \mathcal{H}^m(U) \text{ is the Hausdorff measure.}
\]

An integral current space \(M = (X, d, T)\) is a metric space \((X, d)\) and integral current \(T\) s.t. \(\text{set}(T) = X\) where

\[
\text{set}(T) = \{z \in Z \mid \liminf_{r \to 0} \|T\|(B(z, r))/r^m > 0\}
\]

So it has a countable collection of biLipschitz charts that are oriented and weighted by \(\theta : M \to \mathbb{Z}\)

The mass \(M(U) = \|T\|(U)\) has \(\|T\| = \theta \lambda \mathcal{H}^m\).
Oriented Riemannian Manifolds and Integral Current Spaces

An oriented Riemannian manifold $(M^m, g)$ is a metric space $(M, d_M)$ with a smooth collection of charts

$$d_M(p, q) = \inf \{ L_g(C) : C : [0, 1] \to M, \ C(0) = p, \ C(1) = q \}$$

where $L_g(C) = \int_0^1 g(C'(s), C'(s))^{1/2} \, ds$

$Vol(U) = \mathcal{H}^m(U)$ is the Hausdorff measure.

An integral current space $M = (X, d, T)$ is a metric space $(X, d)$ and integral current $T$ s.t. $\text{set}(T) = X$ where

$$\text{set}(T) = \{ z \in Z \mid \liminf_{r \to 0} \|T\|(B(z, r))/r^m > 0 \}$$

So it has a countable collection of biLipschitz charts that are oriented and weighted by $\theta : M \to \mathbb{Z}$

The mass $\mathcal{M}(U) = \|T\|(U)$ has $\|T\| = \theta \lambda \mathcal{H}^m$.

It might not be connected and might not have any geodesics.
Oriented Riemannian Manifolds and Integral Current Spaces

An oriented Riemannian manifold \((M^m, g)\) is a metric space \((M, d_M)\) with a smooth collection of charts

\[
d_M(p, q) = \inf \left\{ L_g(C) : C : [0, 1] \to M, \ C(0) = p, \ C(1) = q \right\}
\]

where \(L_g(C) = \int_0^1 g(C'(s), C'(s))^{1/2} \, ds\)

\[
\text{Vol}(U) = \mathcal{H}^m(U) \text{ is the Hausdorff measure.}
\]

An integral current space \(M = (X, d, T)\) is a metric space \((X, d)\) and integral current \(T\) s.t. \(\text{set}(T) = X\) where

\[
\text{set}(T) = \left\{ z \in \mathbb{Z} \mid \liminf_{r \to 0} \|T\|(B(z, r))/r^m > 0 \right\}
\]

So it has a countable collection of biLipschitz charts that are oriented and weighted by \(\theta : M \to \mathbb{Z}\)

The mass \(M(U) = \|T\|(U)\) has \(\|T\| = \theta \lambda \mathcal{H}^m\).

It might not be connected and might not have any geodesics.

Its boundary is \(\partial M = (\text{set}(\partial T), d_M, \partial T)\).
Oriented Riemannian Manifolds and Integral Current Spaces

An oriented Riemannian manifold \((M^m, g)\) is a metric space \((M, d_M)\) with a smooth collection of charts
\[
d_M(p, q) = \inf \{ L_g(C) : C : [0, 1] \rightarrow M, \ C(0) = p, \ C(1) = q \}
\]
where
\[
L_g(C) = \int_0^1 g(C'(s), C'(s))^{1/2} \, ds
\]

\[
\text{Vol}(U) = \mathcal{H}^m(U)
\]
is the Hausdorff measure.

An integral current space \(M = (X, d, T)\) is a metric space \((X, d)\) and integral current \(T\) s.t. \(\text{set}(T) = X\) where
\[
\text{set}(T) = \{ z \in Z | \liminf_{r \rightarrow 0} \| T \|(B(z, r))/r^m > 0 \}
\]
So it has a countable collection of biLipschitz charts that are oriented and weighted by \(\theta : M \rightarrow \mathbb{Z}\)

The mass \(\mathbf{M}(U) = \| T \|(U)\) has \(\| T \| = \theta \lambda \mathcal{H}^m\).

It might not be connected and might not have any geodesics.

Its boundary is \(\partial M = (\text{set}(\partial T), d_M, \partial T)\).
An oriented Riemannian manifold \((M^m, g)\) is a metric space \((M, d_M)\) with a smooth collection of charts
\[
d_M(p, q) = \inf \{L_g(C) : C : [0, 1] \to M, C(0) = p, C(1) = q\}
\]
where \(L_g(C) = \int_0^1 g(C'(s), C'(s))^{1/2} \, ds\)
\[
Vol(U) = \mathcal{H}^m(U) \text{ is the Hausdorff measure.}
\]
An integral current space \(M = (X, d, T)\) is a metric space \((X, d)\) and integral current \(T\) s.t. \(\text{set}(T) = X\) where
\[
\text{set}(T) = \{z \in Z | \lim \inf_{r \to 0} \|T\|(B(z, r))/r^m > 0\}
\]
So it has a countable collection of biLipschitz charts
that are oriented and weighted by \(\theta : M \to \mathbb{Z}\)
The mass \(\mathbf{M}(U) = \|T\|(U)\) has \(\|T\| = \theta \lambda \mathcal{H}^m\).
It might not be connected and might not have any geodesics.
Its boundary is \(\partial M = (\text{set}(\partial T), d_M, \partial T)\).
A compact oriented manifold \((M^m, g)\) is an integral current space \((M, d_M, [\![M]\!]\)) with weight \(\theta = 1\) and \(\mathbf{M}(U) = \text{Vol}(U) = \mathcal{H}^m(U)\).
Oriented Riemannian Manifolds and Integral Current Spaces

An oriented Riemannian manifold $(M^m, g)$ is a metric space $(M, d_M)$ with a smooth collection of charts

$$d_M(p, q) = \inf \{ L_g(C) : C : [0, 1] \to M, \ C(0) = p, \ C(1) = q \}$$

where $L_g(C) = \int_0^1 g(C'(s), C'(s))^{1/2} \, ds$

$Vol(U) = \mathcal{H}^m(U)$ is the Hausdorff measure.

An integral current space $M = (X, d, T)$ is a metric space $(X, d)$ and integral current $T$ s.t. $\text{set}(T) = X$ where

$$\text{set}(T) = \{ z \in Z | \liminf_{r \to 0} \| T \|(B(z, r))/r^m > 0 \}$$

So it has a countable collection of biLipschitz charts that are oriented and weighted by $\theta : M \to \mathbb{Z}$

The mass $M(U) = \| T \|(U)$ has $\| T \| = \theta \lambda \mathcal{H}^m$.

It might not be connected and might not have any geodesics.

Its boundary is $\partial M = (\text{set}(\partial T), d_M, \partial T)$.

A compact oriented manifold $(M^m, g)$ is an integral current space $(M, d_M, [[M]])$ with weight $\theta = 1$ and $M(U) = Vol(U) = \mathcal{H}^m(U)$.

Its boundary $(\partial M, d_M, [[\partial M]])$ has the restricted distance $d_M$. 
Sormani-Wenger: Intrinsic Flat Distance

between integral current spaces $M_i^m = (X_i, d_i, T_i)$ is:

$$d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d_{\mathcal{F}}(\varphi_1# T_1, \varphi_2# T_2) \mid \varphi_i : M_i^m \to \mathcal{Z} \right\}$$

where $\varphi# T(f, \pi_1, \ldots, \pi_m) = T(f \circ \varphi, \pi_1 \circ \varphi, \ldots, \pi_m \circ \varphi)$,
Sormani-Wenger: Intrinsic Flat Distance

between integral current spaces $M_i^m = (X_i, d_i, T_i)$ is:

$$d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d^Z_F (\varphi_1\# T_1, \varphi_2\# T_2) \mid \varphi_i : M_i^m \to Z \right\}$$

where $\varphi\# T(f, \pi_1, ..., \pi_m) = T(f \circ \varphi, \pi_1 \circ \varphi, ..., \pi_m \circ \varphi)$, and

where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i^m \to Z$.

Here: $d^Z_F (\varphi_1\# T_1, \varphi_2\# T_2)$ is the Federer-Fleming Flat dist

$$= \inf \left\{ M_{area}(A) + M_{vol}(B) : A + \partial B = \varphi_1\# T_1 - \varphi_2\# T_2 \right\}$$
Sormani-Wenger: Intrinsic Flat Distance

between integral current spaces \( M_i^m = (X_i, d_i, T_i) \) is:

\[
d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d^Z_F(\varphi_1#T_1, \varphi_2#T_2) \mid \varphi_i : M_i^m \to Z \right\}
\]

where \( \varphi#T(f, \pi_1, \ldots, \pi_m) = T(f \circ \varphi, \pi_1 \circ \varphi, \ldots, \pi_m \circ \varphi) \), and
where the infimum is taken over all complete metric spaces, \( Z \),
and over all distance preserving maps \( \varphi_i : M_i^m \to Z \).

Here: \( d^Z_F(\varphi_1#T_1, \varphi_2#T_2) \) is the Federer-Fleming Flat dist

\[
= \inf \left\{ M_{\text{area}}(A) + M_{\text{vol}}(B) : A + \partial B = \varphi_1#T_1 - \varphi_2#T_2 \right\}
\]
Sormani-Wenger: Intrinsic Flat Distance

between integral current spaces \( M_i^m = (X_i, d_i, T_i) \) is:

\[
d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d^Z_F(\varphi_1#T_1, \varphi_2#T_2) \mid \varphi_i : M_i^m \to Z \right\}
\]

where \( \varphi#T(f, \pi_1, ..., \pi_m) = T(f \circ \varphi, \pi_1 \circ \varphi, ..., \pi_m \circ \varphi) \), and
where the infimum is taken over all complete metric spaces, \( Z \),
and over all distance preserving maps \( \varphi_i : M_i^m \to Z \).

Here: \( d^Z_F(\varphi_1#T_1, \varphi_2#T_2) \) is the Federer-Fleming Flat dist

\[
= \inf \left\{ \mathbf{M}_{\text{area}}(\mathbf{A}) + \mathbf{M}_{\text{vol}}(\mathbf{B}) : \mathbf{A} + \partial \mathbf{B} = \varphi_1#T_1 - \varphi_2#T_2 \right\}
\]

**Thm [SW-JDG]:** If \( M_i \) are compact and \( d_{SWIF}(M_1, M_2) = 0 \) then \( \exists \) a current preserving isometry \( F : M_1 \to M_2 \).
Sormani-Wenger: Intrinsic Flat Distance

between integral current spaces $M_i^m = (X_i, d_i, T_i)$ is:

$$d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d_Z^Z (\varphi_1# T_1, \varphi_2# T_2) \mid \varphi_i : M_i^m \to Z \right\}$$

where $\varphi# T(f, \pi_1, ..., \pi_m) = T(f \circ \varphi, \pi_1 \circ \varphi, ..., \pi_m \circ \varphi)$, and

where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i^m \to Z$.

Here: $d_Z^Z (\varphi_1# T_1, \varphi_2# T_2)$ is the Federer-Fleming Flat dist

$$= \inf \left\{ \mathbf{M}_{\text{area}}(\textcolor{green}{A}) + \mathbf{M}_{\text{vol}}(\textcolor{yellow}{B}) : \textcolor{green}{A} + \partial \textcolor{yellow}{B} = \varphi_1# T_1 - \varphi_2# T_2 \right\}$$

Thm [SW-JDG]: If $M_i$ are compact and $d_{SWIF}(M_1, M_2) = 0$ then $\exists$ a current preserving isometry $F : M_1 \to M_2$:

$$d_2(F(p), F(q)) = d_1(p, q) \ \forall p, q \in X_1 \text{ and } F# T_1 = T_2.$$
Sormani-Wenger: Intrinsic Flat Distance

between integral current spaces $M_i^m = (X_i, d_i, T_i)$ is:

$$d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d^Z_F (\varphi_1\#T_1, \varphi_2\#T_2) \mid \varphi_i : M_i^m \to Z \right\}$$

where $\varphi\#T(f, \pi_1, ..., \pi_m) = T(f \circ \varphi, \pi_1 \circ \varphi, ..., \pi_m \circ \varphi)$, and where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i^m \to Z$.

Here: $d^Z_F (\varphi_1\#T_1, \varphi_2\#T_2)$ is the Federer-Fleming Flat dist

$$= \inf \left\{ \text{M}_{\text{area}}(\text{A}) + \text{M}_{\text{vol}}(\text{B}) : \text{A} + \partial \text{B} = \varphi_1\#T_1 - \varphi_2\#T_2 \right\}$$

Thm [SW-JDG]: If $M_i$ are compact and $d_{SWIF}(M_1, M_2) = 0$ then $\exists$ a current preserving isometry $F : M_1 \to M_2$:

$$d_2(F(p), F(q)) = d_1(p, q) \ \forall p, q \in X_1 \text{ and } F\#T_1 = T_2.$$
Sormani-Wenger: Intrinsic Flat Distance

between integral current spaces $M^m_i = (X_i, d_i, T_i)$ is:

$$d_{SWIF}(M^m_1, M^m_2) = \inf \left\{ d^Z_F(\varphi_1\#T_1, \varphi_2\#T_2) \mid \varphi_i : M^m_i \to Z \right\}$$

where $\varphi\#T(f, \pi_1, \ldots, \pi_m) = T(f \circ \varphi, \pi_1 \circ \varphi, \ldots, \pi_m \circ \varphi)$, and

where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M^m_i \to Z$.

Here: $d^Z_F(\varphi_1\#T_1, \varphi_2\#T_2)$ is the Federer-Fleming Flat dist

$$= \inf \left\{ M_{area}(A) + M_{vol}(B) : A + \partial B = \varphi_1\#T_1 - \varphi_2\#T_2 \right\}$$

**Thm [SW-JDG]:** If $M_i$ are compact and $d_{SWIF}(M_1, M_2) = 0$ then $\exists$ a current preserving isometry $F : M_1 \to M_2$:

$$d_2(F(p), F(q)) = d_1(p, q) \ \forall p, q \in X_1 \text{ and } F\#T_1 = T_2.$$ 

**Pf:** Show inf achieved: $\exists \varphi_i : M_i \to Z$ s.t. $\varphi_1\#T_1 = \varphi_2\#T_2$. 
Sormani-Wenger: Intrinsic Flat Distance

between integral current spaces \( M_i^m = (X_i, d_i, T_i) \) is:

\[
d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d^Z_F(\varphi_1#T_1, \varphi_2#T_2) \mid \varphi_i : M_i^m \to Z \right\}
\]

where \( \varphi#T(f, \pi_1, …, \pi_m) = T(f \circ \varphi, \pi_1 \circ \varphi, …, \pi_m \circ \varphi) \), and where the infimum is taken over all complete metric spaces, \( Z \), and over all distance preserving maps \( \varphi_i : M_i^m \to Z \).

Here: \( d^Z_F(\varphi_1#T_1, \varphi_2#T_2) \) is the Federer-Fleming Flat dist

\[
= \inf \left\{ \mathbf{M}_{\text{area}}(\varphi_1#T_1) + \mathbf{M}_{\text{vol}}(\varphi_2#T_2) : \varphi_1#T_1 - \varphi_2#T_2 \right\}
\]

**Thm [SW-JDG]:** If \( M_i \) are compact and \( d_{SWIF}(M_1, M_2) = 0 \)

then \( \exists \) a current preserving isometry \( F : M_1 \to M_2 : \)

\[
d_2(F(p), F(q)) = d_1(p, q) \ \forall p, q \in X_1 \ \text{and} \ F#T_1 = T_2.
\]

**Pf:** Show inf achieved: \( \exists \varphi_i : M_i \to Z \) s.t. \( \varphi_1#T_1 = \varphi_2#T_2 \).

So set(\( \varphi_1#T_1 \)) = set(\( \varphi_2#T_2 \)) and \( F = \varphi_2^{-1} \circ \varphi_1 \) is defined.
Sormani-Wenger: Intrinsic Flat Distance

between integral current spaces \( M_i^m = (X_i, d_i, T_i) \) is:

\[
d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d_Z^2(\varphi_1\# T_1, \varphi_2\# T_2) \mid \varphi_i : M_i^m \to Z \right\}
\]

where \( \varphi\# T(f, \pi_1, ..., \pi_m) = T(f \circ \varphi, \pi_1 \circ \varphi, ..., \pi_m \circ \varphi) \), and

where the infimum is taken over all complete metric spaces, \( Z \),

and over all distance preserving maps \( \varphi_i : M_i^m \to Z \).

Here: \( d_Z^2(\varphi_1\# T_1, \varphi_2\# T_2) \) is the Federer-Fleming Flat dist

\[
= \inf \left\{ M_{\text{area}}(\text{A}) + M_{\text{vol}}(\text{B}) : \text{A} + \partial \text{B} = \varphi_1\# T_1 - \varphi_2\# T_2 \right\}
\]

Thm [SW-JDG]: If \( M_i \) are compact and \( d_{SWIF}(M_1, M_2) = 0 \)

then \( \exists \) a current preserving isometry \( F : M_1 \to M_2 : \)

\( d_2(F(p), F(q)) = d_1(p, q) \ \forall p, q \in X_1 \) and \( F\# T_1 = T_2 \).

Pf: Show inf achieved: \( \exists \varphi_i : M_i \to Z \) s.t. \( \varphi_1\# T_1 = \varphi_2\# T_2 \).

So \( \text{set}(\varphi_1\# T_1) = \text{set}(\varphi_2\# T_2) \) and \( F = \varphi_2^{-1} \circ \varphi_1 \) is defined.
The zero space $0^m = (\emptyset, 0, 0)$ is an integral current space

$$d_{SWIF}(M^m_1, 0^m) = \inf \left\{ d^Z_F (\varphi_1\# T_1, \varphi_2\#0) \mid \varphi_i : M^m_i \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all dist. pres. maps $\varphi_i : M^m_i \to Z$. ($\varphi_1$ trivial)

Here: $d^Z_F (\varphi_1\# T_1, \varphi_2\#0)$ is the Federer-Fleming Flat dist

$$= \inf \left\{ \mathbf{M}_{area}(A) + \mathbf{M}_{vol}(B) : A + \partial B = \varphi_1\# T_1 - \varphi_2\#0 \right\}$$
The zero space $0^m = (\emptyset, 0, 0)$ is an integral current space

$$d_{SWIF}(M_1^m, 0^m) = \inf \left\{ d^Z_F(\varphi_1\# T_1, \varphi_2\# 0) \mid \varphi_i : M_i^m \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all dist. pres. maps $\varphi_i : M_i^m \to Z$. ($\varphi_1$ trivial)

Here: $d^Z_F(\varphi_1\# T_1, \varphi_2\# 0)$ is the Federer-Fleming Flat dist

$$= \inf \left\{ \mathrm{M}_{area}(A) + \mathrm{M}_{vol}(B) : A + \partial B = \varphi_1\# T_1 - \varphi_2\# 0 \right\}$$

**Thm [SW]:** If $M$ Riemannian then $d_{SWIF}(M^m, 0^m) \leq \mathrm{Vol}(M)$. 
The zero space $0^m = (\emptyset, 0, 0)$ is an integral current space

$$d_{SWIF}(M^m_1, 0^m) = \inf \left\{ d_{F}^{Z} (\varphi_1# T_1, \varphi_2#0) \mid \varphi_i : M_i^m \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all dist. pres. maps $\varphi_i : M_i^m \to Z$. ($\varphi_1$ trivial)

Here: $d_{F}^{Z} (\varphi_1# T_1, \varphi_2#0)$ is the Federer-Fleming Flat dist

$$= \inf \left\{ M_{\text{area}}(A) + M_{\text{vol}}(B) : A + \partial B = \varphi_1# T_1 - \varphi_2#0 \right\}$$

**Thm [SW]:** If $M$ Riemannian then $d_{SWIF}(M^m, 0^m) \leq \text{Vol}(M)$.

**Pf:** Take $Z = M$, $\varphi_1 = id$, $A = id_#[[M]] = [[M]]$, and $B = 0$. □
The zero space \(0^m = (\emptyset, 0, 0)\) is an integral current space

\[
d_{SWIF}(M_1^m, 0^m) = \inf \left\{ d^Z_F (\varphi_1# T_1, \varphi_2#0) \mid \varphi_i : M_i^m \rightarrow Z \right\}
\]

where the infimum is taken over all complete metric spaces, \(Z\), and over all dist. pres. maps \(\varphi_i : M_i^m \rightarrow Z\). (\(\varphi_1\) trivial)

Here: \(d^Z_F (\varphi_1# T_1, \varphi_2#0)\) is the Federer-Fleming Flat dist

\[
= \inf \left\{ \begin{array}{c}
M_{\text{area}}(A) + M_{\text{vol}}(B) : A + \partial B = \varphi_1# T_1 - \varphi_2#0
\end{array} \right\}
\]

**Thm [SW]:** If \(M\) Riemannian then \(d_{SWIF}(M^m, 0^m) \leq \text{Vol}(M)\).

**Pf:** Take \(Z = M\), \(\varphi_1 = id\), \(A = id#[[M]] = [[M]]\), and \(B = 0\). □

**Example:** \(d_{SWIF}(S^m, 0^m) \leq \text{Vol}(S^{m+1})/2\).
The zero space $0^m = (\emptyset, 0, 0)$ is an integral current space

$$d_{SWIF}(M_1^m, 0^m) = \inf \left\{ d_Z^F(\varphi_1\# T_1, \varphi_2\# 0) \mid \varphi_i : M_i^m \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all dist. pres. maps $\varphi_i : M_i^m \to Z$. ($\varphi_1$ trivial)

Here: $d_Z^F(\varphi_1\# T_1, \varphi_2\# 0)$ is the Federer-Fleming Flat dist

$$= \inf \left\{ \text{M}_{\text{area}}(A) + \text{M}_{\text{vol}}(B) : A + \partial B = \varphi_1\# T_1 - \varphi_2\# 0 \right\}$$

**Thm [SW]:** If $M$ Riemannian then $d_{SWIF}(M^m, 0^m) \leq \text{Vol}(M)$.

**Pf:** Take $Z = M$, $\varphi_1 = \text{id}$, $A = \text{id}\#[[M]] = [[M]]$, and $B = 0$. □

**Example:** $d_{SWIF}(S^m, 0^m) \leq \text{Vol}(S^{m+1})/2$.

**Pf:** Take $Z = S^{m+1}$ so $\varphi_1 : S^m \to \text{Equator} \subset S^{m+1}$ is dist pres.
The zero space $0^m = (\emptyset, 0, 0)$ is an integral current space

$$d_{SWIF}(M_1^m, 0^m) = \inf \left\{ d^Z_F (\varphi_1# T_1, \varphi_2#0) \mid \varphi_i : M_i^m \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all dist. pres. maps $\varphi_i : M_i^m \to Z$. ($\varphi_1$ trivial)

Here: $d^Z_F (\varphi_1# T_1, \varphi_2#0)$ is the Federer-Fleming Flat dist

$$= \inf \left\{ \text{M}_{\text{area}}(A) + \text{M}_{\text{vol}}(B) : A + \partial B = \varphi_1# T_1 - \varphi_2#0 \right\}$$

**Thm [SW]:** If $M$ Riemannian then $d_{SWIF}(M^m, 0^m) \leq \text{Vol}(M)$.

**Pf:** Take $Z = M$, $\varphi_1 = id$, $A = id#[[M]] = [[M]]$, and $B = 0$. □

**Example:** $d_{SWIF} (S^m, 0^m) \leq \text{Vol}(S^{m+1})/2$.

**Pf:** Take $Z = S^{m+1}$ so $\varphi_1 : S^m \to \text{Equator} \subset S^{m+1}$ is dist pres. (Note $Z = D^{m+1}$ fails to have dist. pres $\varphi_1 : S^m \to Z$).
The zero space $0^m = (\emptyset, 0, 0)$ is an integral current space

$$d_{SWIF}(M_1^m, 0^m) = \inf \left\{ d_Z^F(\varphi_1\# T_1, \varphi_2\#0) \mid \varphi_i : M_i^m \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all dist. pres. maps $\varphi_i : M_i^m \to Z$. ($\varphi_1$ trivial)

Here: $d_Z^F(\varphi_1\# T_1, \varphi_2\#0)$ is the Federer-Fleming Flat dist

$$= \inf \left\{ M_{\text{area}}(A) + M_{\text{vol}}(B) : A + \partial B = \varphi_1\# T_1 - \varphi_2\#0 \right\}$$

**Thm [SW]:** If $M$ Riemannian then $d_{SWIF}(M^m, 0^m) \leq \text{Vol}(M)$.

**Pf:** Take $Z = M$, $\varphi_1 = id$, $A = id_{\#}[[M]] = [[M]]$, and $B = 0$. □

**Example:** $d_{SWIF}(S^m, 0^m) \leq \text{Vol}(S^{m+1})/2$.

**Pf:** Take $Z = S^{m+1}$ so $\varphi_1 : S^m \to \text{Equator} \subset S^{m+1}$ is dist pres. (Note $Z = D^{m+1}$ fails to have dist. pres $\varphi_1 : S^m \to Z$).

Take $B = [[S^{m+1}_+]]$ so $\partial B = \varphi_1\#[[S^m]]$ and $A = 0$. □
Implications of SWIF Convergence [SW-JDG]

**Defn:** For any pair of integral current spaces,

\[ d_{SWIF}(M^m_1, M^m_2) = \inf \left\{ d^Z_F(\varphi_1\#T_1, \varphi_2\#T_2) \mid \varphi_i : M^m_i \to Z \right\} \]

where the inf over complete \( Z \) and dist. pres. \( \varphi_i : M^m_i \to Z \).

**Thm:** The infimum is achieved, so we can choose

\[ Z' = \text{set}(A) \cup \text{set}(B) \]

which is separable and rectifiable.
Implications of SWIF Convergence [SW-JDG]

**Defn:** For any pair of integral current spaces,

\[
d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d_F^Z (\varphi_1# T_1, \varphi_2# T_2) \mid \varphi_i : M_i^m \to Z \right\}
\]

where the inf over complete \( Z \) and dist. pres. \( \varphi_i : M_i^m \to Z \).

**Thm:** The infimum is achieved, so we can choose

\( Z' = \text{set}(A) \cup \text{set}(B) \) which is separable and rectifiable.

**Thm:** If \( M_j = (X_j, d_j, T_j) \xrightarrow{SWIF} M_0 = (X_0, d_0, T_0) \)
Implications of SWIF Convergence [SW-JDG]

**Defn:** For any pair of integral current spaces,

$$d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d^Z_F (\varphi_1#T_1, \varphi_2#T_2) \mid \varphi_i : M_i^m \to Z \right\}$$

where the inf over complete $Z$ and dist. pres. $\varphi_i : M_i^m \to Z$.

**Thm:** The infimum is achieved, so we can choose $Z' = \text{set}(A) \cup \text{set}(B)$ which is separable and rectifiable.

**Thm:** If $M_j = (X_j, d_j, T_j) \xrightarrow{SWIF} M_0 = (X_0, d_0, T_0)$ then $\exists Z'_j$ s.t. $d_{SWIF}(M_j, M_0) = d^Z_j F (\varphi_j#T_j, \varphi_{0,j#}T_0)$
Implications of SWIF Convergence [SW-JDG]

**Defn:** For any pair of integral current spaces,

\[ d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d_Z^F(\varphi_1#T_1, \varphi_2#T_2) \mid \varphi_i : M_i^m \to Z \right\} \]

where the inf over complete \( Z \) and dist. pres. \( \varphi_i : M_i^m \to Z \).

**Thm:** The infimum is achieved, so we can choose \( Z' = \text{set}(A) \cup \text{set}(B) \) which is separable and rectifiable.

**Thm:** If \( M_j = (X_j, d_j, T_j) \xrightarrow{SWIF} M_0 = (X_0, d_0, T_0) \)

then \( \exists Z'_j \) s.t. \( d_{SWIF}(M_j, M_0) = d_{Z_j}^F(\varphi_j#T_j, \varphi_{0,j}#T_0) \)

which we can glue along the images \( \varphi_{0,j}(M_0) \) to show
Implications of SWIF Convergence [SW-JDG]

**Defn:** For any pair of integral current spaces,

\[ d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d_Z^F (\varphi_1\# T_1, \varphi_2\# T_2) \mid \varphi_i : M_i^m \to Z \right\} \]

where the inf over complete \( Z \) and dist. pres. \( \varphi_i : M_i^m \to Z \).

**Thm:** The infimum is achieved, so we can choose \( Z' = \text{set}(A) \cup \text{set}(B) \) which is separable and rectifiable.

**Thm:** If \( M_j = (X_j, d_j, T_j) \xrightarrow{SWIF} M_0 = (X_0, d_0, T_0) \)

then \( \exists Z_j' \) s.t. \( d_{SWIF}(M_j, M_0) = d_{Z_j'}^F (\varphi_j\# T_j, \varphi_{0,j}\# T_0) \)

which we can glue along the images \( \varphi_{0,j}(M_0) \) to show \( \exists \) complete separable \( Z \) and dist. pres. \( \varphi_j : X_j \to Z \)

s.t. \( d_Z^F (\varphi_j\# T_j, \varphi_{0}\# T_0) \to 0 \) and \( \varphi_j\# T_j(\omega) \to \varphi_{0}\# T_0(\omega) \).
Implications of SWIF Convergence [SW-JDG]

**Defn:** For any pair of integral current spaces,

\[ d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d^Z_F(\varphi_1#T_1, \varphi_2#T_2) \mid \varphi_i : M_i^m \to Z \right\} \]

where the inf over complete \( Z \) and dist. pres. \( \varphi_i : M_i^m \to Z \).

**Thm:** The infimum is achieved, so we can choose \( Z' = \text{set}(A) \cup \text{set}(B) \) which is separable and rectifiable.

**Thm:** If \( M_j = (X_j, d_j, T_j) \overset{SWIF}{\longrightarrow} M_0 = (X_0, d_0, T_0) \)

then \( \exists Z'_j \) s.t. \( d_{SWIF}(M_j, M_0) = d^Z'_F(\varphi_j#T_j, \varphi_0.j#T_0) \)

which we can glue along the images \( \varphi_{0,j}(M_0) \) to show

\( \exists \) complete separable \( Z \) and dist. pres. \( \varphi_j : X_j \to Z \)

s.t. \( d^Z_F(\varphi_j#T_j, \varphi_0#T_0) \to 0 \) and \( \varphi_j#T_j(\omega) \to \varphi_0#T_0(\omega) \).

Thus by Ambrosio-Kirchheim Theory:
Implications of SWIF Convergence [SW-JDG]

**Defn:** For any pair of integral current spaces, 

\[ d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d_F^Z(\varphi_1^# T_1, \varphi_2^# T_2) \mid \varphi_i : M_i^m \to Z \right\} \]

where the inf over complete \( Z \) and dist. pres. \( \varphi_i : M_i^m \to Z \).

**Thm:** The infimum is achieved, so we can choose \( Z' = \text{set}(A) \cup \text{set}(B) \) which is separable and rectifiable.

**Thm:** If \( M_j = (X_j, d_j, T_j) \xrightarrow{SWIF} M_0 = (X_0, d_0, T_0) \)

then \( \exists Z_j' \) s.t. \( d_{SWIF}(M_j, M_0) = d_F^{Z_j'}(\varphi_j^# T_j, \varphi_{0,j}^# T_0) \)

which we can glue along the images \( \varphi_{0,j}(M_0) \) to show

\( \exists \) complete separable \( Z \) and dist. pres. \( \varphi_j : X_j \to Z \)

s.t. \( d_F^Z(\varphi_j^# T_j, \varphi_0^# T_0) \to 0 \) and \( \varphi_j^# T_j(\omega) \to \varphi_0^# T_0(\omega) \).

**Thus by Ambrosio-Kirchheim Theory:**

\[ M_j \xrightarrow{SWIF} M_\infty \implies \partial M_j \xrightarrow{SWIF} \partial M_\infty \]
Implications of SWIF Convergence [SW-JDG]

**Defn:** For any pair of integral current spaces,

\[ d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d_Z^F(\varphi_1\# T_1, \varphi_2\# T_2) \mid \varphi_i : M_i^m \to Z \right\} \]

where the inf over complete \( Z \) and dist. pres. \( \varphi_i : M_i^m \to Z \).

**Thm:** The infimum is achieved, so we can choose

\( Z' = \text{set}(A) \cup \text{set}(B) \) which is separable and rectifiable.

**Thm:** If \( M_j = (X_j, d_j, T_j) \xrightarrow{SWIF} M_0 = (X_0, d_0, T_0) \)

then \( \exists Z'_j \) s.t. \( d_{SWIF}(M_j, M_0) = d_{\text{ref}}(\varphi_j\# T_j, \varphi_0\# T_0) \)

which we can glue along the images \( \varphi_{0,j}(M_0) \) to show

\( \exists \) complete separable \( Z \) and dist. pres. \( \varphi_j : X_j \to Z \)

s.t. \( d_Z^F(\varphi_j\# T_j, \varphi_0\# T_0) \to 0 \) and \( \varphi_j\# T_j(\omega) \to \varphi_0\# T_0(\omega) \).

**Thus by Ambrosio-Kirchheim Theory:**

\[ M_j \xrightarrow{SWIF} M_\infty \implies \partial M_j \xrightarrow{SWIF} \partial M_\infty \]

\[ M_j \xrightarrow{SWIF} M_\infty \implies \lim \inf_{j \to \infty} M(M_j) \geq M(M_\infty) \]
**SWIF Compactness Theorems**

**Thm [SW]:** If $M_j \xrightarrow{GH} M_{GH}$ and $\text{Vol}(M_j) \leq V_0$ and $\text{Vol}(\partial M_j) \leq A_0$ then $\exists M_{j_k} \xrightarrow{\text{SWIF}} M_{\text{SWIF}}$ where $M_{\text{SWIF}} \subset M_{GH}$ or $M_{\text{SWIF}} = 0$.

*How do we know which regions disappear? Use Filling Volumes!*
SWIF Compactness Theorems

**Thm [SW]:** If $M_j \xrightarrow{GH} M_{GH}$ and $\text{Vol}(M_j) \leq V_0$ and $\text{Vol}(\partial M_j) \leq A_0$ then $\exists M_{j_k} \xrightarrow{\text{SWIF}} M_{\text{SWIF}}$ where $M_{\text{SWIF}} \subset M_{GH}$ or $M_{\text{SWIF}} = 0$.

**Proof:** By Gromov’s Compactness Thm, $\exists$ compact $Z$ and dist pres maps $\varphi_j : M_j \to Z$ s.t. $d_Z^H(\varphi_j(M_j), \varphi_{GH}(M_{GH})) \to 0$. How do we know which regions disappear? Use Filling Volumes!
SWIF Compactness Theorems

**Thm [SW]:** If \( M_j \xrightarrow{GH} M_{GH} \) and \( \text{Vol}(M_j) \leq V_0 \) and \( \text{Vol}(\partial M_j) \leq A_0 \) then \( \exists M_{j_k} \xrightarrow{SWIF} M_{SWIF} \) where \( M_{SWIF} \subset M_{GH} \) or \( M_{SWIF} = 0 \).

**Proof:** By Gromov’s Compactness Thm, \( \exists \) compact \( Z \) and dist pres maps \( \varphi_j : M_j \to Z \) s.t. \( d^Z_H(\varphi_j(M_j), \varphi_{GH}(M_{GH})) \to 0 \).
By Ambrosio-Kirchheim Compactness: \( \exists \) subseq \( \varphi_{j\#} T_j \to T_\infty \).
**SWIF Compactness Theorems**

**Thm [SW]:** If $M_j \xrightarrow{GH} M_{GH}$ and $\text{Vol}(M_j) \leq V_0$ and $\text{Vol}(\partial M_j) \leq A_0$ then $\exists M_{j_k} \xrightarrow{SWIF} M_{SWIF}$ where $M_{SWIF} \subset M_{GH}$ or $M_{SWIF} = 0$.

**Proof:** By Gromov’s Compactness Thm, $\exists$ compact $Z$ and dist pres maps $\varphi_j : M_j \rightarrow Z$ s.t. $d^Z_H(\varphi_j(M_j), \varphi_{GH}(M_{GH})) \rightarrow 0$. By Ambrosio-Kirchheim Compactness: $\exists$ subseq $\varphi_{j\#} T_j \rightarrow T_\infty$. $\text{set}(T_\infty) \subset \varphi_{GH}(M_{GH}) \subset Z$. 
**SWIF Compactness Theorems**

**Thm [SW]:** If $M_j \xrightarrow{GH} M_{GH}$ and $\text{Vol}(M_j) \leq V_0$ and $\text{Vol}(\partial M_j) \leq A_0$ then $\exists M_{j_k} \xrightarrow{SWIF} M_{SWIF}$ where $M_{SWIF} \subset M_{GH}$ or $M_{SWIF} = 0$.

**Proof:** By Gromov’s Compactness Thm, $\exists$ compact $Z$ and dist pres maps $\varphi_j : M_j \to Z$ s.t. $d^Z_H(\varphi_j(M_j), \varphi_{GH}(M_{GH})) \to 0$. By Ambrosio-Kirchheim Compactness: $\exists$ subseq $\varphi_{j\#} T_j \to T_\infty$. set$(T_\infty) \subset \varphi_{GH}(M_{GH}) \subset Z$. Let $M_{SWIF} = (\text{set}(T_\infty), d_Z, T_\infty)$ \(\square\).
**SWIF Compactness Theorems**

**Thm [SW]:** If $M_j \xrightarrow{GH} M_{GH}$ and $\text{Vol}(M_j) \leq V_0$ and $\text{Vol}(\partial M_j) \leq A_0$
then $\exists M_{j,k} \xrightarrow{SWIF} M_{SWIF}$ where $M_{SWIF} \subset M_{GH}$ or $M_{SWIF} = 0$.

**Proof:**
By Gromov’s Compactness Thm, $\exists$ compact $Z$ and dist pres maps $\varphi_j : M_j \to Z$ s.t. $d^Z_H(\varphi_j(M_j), \varphi_{GH}(M_{GH})) \to 0$.
By Ambrosio-Kirchheim Compactness: $\exists$ subseq $\varphi_{j#} T_j \to T_\infty$.
$\text{set}(T_\infty) \subset \varphi_{GH}(M_{GH}) \subset Z$. Let $M_{SWIF} = (\text{set}(T_\infty), d^Z, T_\infty)$ $\square$.

**Wenger Compactness Thm:** If $\text{Diam}(M_j) \leq D$ and $M(M_j) \leq V$ and $M(\partial M_j) \leq A_0$ then $\exists M_{j,k} \xrightarrow{SWIF} M_{SWIF}$ possibly $M_{SWIF} = 0$. 

---

**Diagrams:**
- SWIF Compactness
- GH Compactness
- Wenger Compactness
SWIF Compactness Theorems

**Thm [SW]:** If \( M_j \xrightarrow{GH} M_{GH} \) and \( \text{Vol}(M_j) \leq V_0 \) and \( \text{Vol}(\partial M_j) \leq A_0 \) then \( \exists M_{j_k} \xrightarrow{SWIF} M_{SWIF} \) where \( M_{SWIF} \subset M_{GH} \) or \( M_{SWIF} = 0 \).

**Proof:** By Gromov’s Compactness Thm, \( \exists \) compact \( Z \) and dist pres maps \( \varphi_j : M_j \to Z \) s.t. \( d^Z_H(\varphi_j(M_j), \varphi_{GH}(M_{GH})) \to 0 \).

By Ambrosio-Kirchheim Compactness: \( \exists \) subseq \( \varphi_j#T_j \to T_\infty \).

set(\( T_\infty \)) \( \subset \varphi_{GH}(M_{GH}) \subset Z \). Let \( M_{SWIF} = (\text{set}(T_\infty), d_Z, T_\infty) \). □

**Wenger Compactness Thm:** If \( \text{Diam}(M_j) \leq D \) and \( \text{M}(M_j) \leq V \) and \( \text{M}(\partial M_j) \leq A_0 \) then \( \exists M_{j_k} \xrightarrow{SWIF} M_{SWIF} \) possibly \( M_{SWIF} = 0 \).
SWIF Compactness Theorems

**Thm [SW]:** If $M_j \xrightarrow{\text{GH}} M_{GH}$ and $\text{Vol}(M_j) \leq V_0$ and $\text{Vol}(\partial M_j) \leq A_0$ then $\exists M_{j_k} \xrightarrow{\text{SWIF}} M_{SWIF}$ where $M_{SWIF} \subset M_{GH}$ or $M_{SWIF} = 0$.

**Proof:** By Gromov’s Compactness Thm, $\exists$ compact $Z$ and dist pres maps $\varphi_j : M_j \to Z$ s.t. $d_Z^H(\varphi_j(M_j), \varphi_{GH}(M_{GH})) \to 0$. By Ambrosio-Kirchheim Compactness: $\exists$ subseq $\varphi_j \# T_j \to T_\infty$. $\text{set}(T_\infty) \subset \varphi_{GH}(M_{GH}) \subset Z$. Let $M_{SWIF} = (\text{set}(T_\infty), d_Z, T_\infty)$ $\square$.

**Wenger Compactness Thm:** If $\text{Diam}(M_j) \leq D$ and $M(M_j) \leq V$ and $M(\partial M_j) \leq A_0$ then $\exists M_{j_k} \xrightarrow{\text{SWIF}} M_{SWIF}$ possibly $M_{SWIF} = 0$. How do we know which regions disappear?
**Thm [SW]:** If $M_j \overset{GH}{\to} M_{GH}$ and $\text{Vol}(M_j) \leq V_0$ and $\text{Vol}(\partial M_j) \leq A_0$ then $\exists M_{j_k} \overset{SWIF}{\to} M_{SWIF}$ where $M_{SWIF} \subset M_{GH}$ or $M_{SWIF} = 0$.

**Proof:** By Gromov’s Compactness Thm, $\exists$ compact $Z$ and dist pres maps $\varphi_j : M_j \to Z$ s.t. $d^Z_H(\varphi_j(M_j), \varphi_{GH}(M_{GH})) \to 0$. By Ambrosio-Kirchheim Compactness: $\exists$ subseq $\varphi_{j\#} T_j \to T_\infty$. $\set{T_\infty} \subset \varphi_{GH}(M_{GH}) \subset Z$. Let $M_{SWIF} = (\set{T_\infty}, d_Z, T_\infty)$ $\square$.

**Wenger Compactness Thm:** If $\text{Diam}(M_j) \leq D$ and $M(M_j) \leq V$ and $M(\partial M_j) \leq A_0$ then $\exists M_{j_k} \overset{SWIF}{\to} M_{SWIF}$ possibly $M_{SWIF} = 0$.

How do we know which regions disappear? Use Filling Volumes!
Adapting Gromov’s Filling Volume [Portegies-Sormani]:

\[ \text{FillVol}(M^m) = \inf \{ M(N^{n+1}) \mid \partial N^{n+1} = M^m \} \]

where the inf is over integral current spaces \( N^{n+1} = (X_N, d_N, T_N) \)
such that \( \exists \) current preserving isometry \( F : M^m \to \partial N^{n+1} \).
Adapting Gromov’s Filling Volume [Portegies-Sormani]:

\[
\text{FillVol}(M^m) = \inf \{ M(N^{n+1}) \mid \partial N^{n+1} = M^m \}
\]

where the inf is over integral current spaces \( N^{n+1} = (X_N, d_N, T_N) \) such that \( \exists \) current preserving isometry \( F : M^m \rightarrow \partial N^{n+1} \).

Recall \( \partial N = (\text{set}(\partial T_N), d_N, \partial T_N) \) has the restricted distance \( d_N \) so \( d_N(F(p), F(q)) = d_M(p, q) \forall p, q \in X_M \) and \( F_\# \partial T_N = T_M \).
Adapting Gromov’s Filling Volume [Portegies-Sormani]:

\[
\text{FillVol}(M^m) = \inf \{ M(N^{n+1}) \mid \partial N^{n+1} = M^m \}
\]

where the inf is over integral current spaces \( N^{n+1} = (X_N, d_N, T_N) \) such that \( \exists \) current preserving isometry \( F : M^m \to \partial N^{n+1} \).

Recall \( \partial N = (\text{set}(\partial T_N), d_N, \partial T_N) \) has the restricted distance \( d_N \) so \( d_N(F(p), F(q)) = d_M(p, q) \forall p, q \in X_M \) and \( F \# \partial T_N = T_M \).

**Example:** \( \text{FillVol}((S^m, d_{S^m}, [[S^m]])) \leq \text{Vol}(S^{m+1})/2 \).
Adapting Gromov’s Filling Volume [Portegies-Sormani]:

\[ \text{FillVol}(M^m) = \inf \{ M(N^{n+1}) \mid \partial N^{n+1} = M^m \} \]

where the inf is over integral current spaces \( N^{n+1} = (X_N, d_N, T_N) \) such that \( \exists \) current preserving isometry \( F : M^m \to \partial N^{n+1} \).

Recall \( \partial N = (\text{set}(\partial T_N), d_N, \partial T_N) \) has the restricted distance \( d_N \) so \( d_N(F(p), F(q)) = d_M(p, q) \ \forall p, q \in X_M \) and \( F_\# \partial T_N = T_M \).

**Example:** \( \text{FillVol}((S^m, d_{S^m}, [[S^m]])) \leq \text{Vol}(S^{m+1})/2. \)

**Pf:** Take \( N = S^{m+1}_+ \) so \( F : S^m \to \text{Equator} \subset S^{m+1}_+ \) is dist pres □
Adapting Gromov’s Filling Volume [Portegies-Sormani]:

\[ \text{FillVol}(\mathcal{M}^m) = \inf \{ \mathcal{M}(\mathcal{N}^{n+1}) | \partial \mathcal{N}^{n+1} = \mathcal{M}^m \} \]

where the inf is over integral current spaces \( \mathcal{N}^{n+1} = (X_N, d_N, T_N) \) such that \( \exists \) current preserving isometry \( F : \mathcal{M}^m \to \partial \mathcal{N}^{n+1} \).

Recall \( \partial \mathcal{N} = (\text{set}(\partial T_N), d_N, \partial T_N) \) has the restricted distance \( d_N \) so \( d_N(F(p), F(q)) = d_M(p, q) \forall p, q \in X_M \) and \( F_\# \partial T_N = T_M \).

**Example:** \( \text{FillVol}((S^m, d_{S^m}, [[S^m]])) \leq \text{Vol}(S^{m+1})/2 \).

**Pf:** Take \( N = S^{m+1}_+ \) so \( F : S^m \to \text{Equator} \subset S^{m+1}_+ \) is dist pres \( \square \)

**Open:** Is \( \text{FillVol}((S^m, d_{S^m}, [[S^m]])) = \text{Vol}(S^{m+1})/2 \)? Pu Conj
Adapting Gromov’s Filling Volume [Portegies-Sormani]:

\[
\text{FillVol}(M^m) = \inf \{ \text{M}(N^{n+1}) \mid \partial N^{n+1} = M^m \}
\]

where the inf is over integral current spaces \( N^{n+1} = (X_N, d_N, T_N) \)
such that \( \exists \) current preserving isometry \( F : M^m \rightarrow \partial N^{n+1} \).

Recall \( \partial N = (\text{set}(\partial T_N), d_N, \partial T_N) \) has the restricted distance \( d_N \)
so \( d_N(F(p), F(q)) = d_M(p, q) \forall p, q \in X_M \) and \( F_{\#} \partial T_N = T_M \).

**Example:** \( \text{FillVol}((S^m, d_{S^m}, [[S^m]])) \leq \text{Vol}(S^{m+1})/2 \).

**Pf:** Take \( N = S_{m+1}^+ \) so \( F : S^m \rightarrow \text{Equator} \subset S_{m+1}^+ \) is dist pres \( \Box \)

**Open:** Is \( \text{FillVol}((S^m, d_{S^m}, [[S^m]])) = \text{Vol}(S^{m+1})/2 \)? Pu Conj

**Example:** \( \text{FillVol}((S^m, d_{D^{m+1}}, [[S^m]])) \leq \text{Vol}(D^{m+1}) \).
Adapting Gromov’s Filling Volume [Portegies-Sormani]:

\[ \text{FillVol}(M^m) = \inf \left\{ M(N^{n+1}) \mid \partial N^{n+1} = M^m \right\} \]

where the inf is over integral current spaces \( N^{n+1} = (X_N, d_N, T_N) \) such that \( \exists \) current preserving isometry \( F : M^m \to \partial N^{n+1} \).

Recall \( \partial N = (\text{set}(\partial T_N), d_N, \partial T_N) \) has the restricted distance \( d_N \) so \( d_N(F(p), F(q)) = d_M(p, q) \forall p, q \in X_M \) and \( F_# \partial T_N = T_M \).

**Example:** \( \text{FillVol}((\mathbb{S}^m, d_{\mathbb{S}^m}, [[\mathbb{S}^m]])) \leq \text{Vol}(\mathbb{S}^{m+1})/2 \).

**Pf:** Take \( N = \mathbb{S}^{m+1}_+ \) so \( F : \mathbb{S}^m \to \text{Equator} \subset \mathbb{S}^{m+1}_+ \) is dist pres \( \square \)

**Open:** Is \( \text{FillVol}((\mathbb{S}^m, d_{\mathbb{S}^m}, [[\mathbb{S}^m]])) = \text{Vol}(\mathbb{S}^{m+1})/2 \)? Pu Conj

**Example:** \( \text{FillVol}((\mathbb{S}^m, d_{\mathbb{D}^m+1}, [[\mathbb{S}^m]])) \leq \text{Vol}(\mathbb{D}^{m+1}) \).

**Pf:** Take \( N = \mathbb{D}^{m+1}_+ \) so \( F : \mathbb{S}^m \to \partial \mathbb{D}^{m+1} \subset \mathbb{D}^{m+1} \) is dist pres \( \square \)
Adapting Gromov’s Filling Volume [Portegies-Sormani]:

\[
\text{FillVol}(M^m) = \inf \{ M(N^{n+1}) \mid \partial N^{n+1} = M^m \}
\]

where the inf is over integral current spaces \( N^{n+1} = (X_N, d_N, T_N) \) such that \( \exists \) current preserving isometry \( F : M^m \to \partial N^{n+1} \).

Recall \( \partial N = (\text{set}(\partial T_N), d_N, \partial T_N) \) has the restricted distance \( d_N \) so \( d_N(F(p), F(q)) = d_M(p, q) \forall p, q \in X_M \) and \( F_\# \partial T_N = T_M \).

**Example:** \( \text{FillVol}((S^m, d_{S^m}, [[S^m]])) \leq \text{Vol}(S^{m+1})/2 \).

**Pf:** Take \( N = S^{m+1}_+ \) so \( F : S^m \to \text{Equator} \subset S^{m+1}_+ \) is dist pres \( \square \)

**Open:** Is \( \text{FillVol}((S^m, d_{S^m}, [[S^m]])) = \text{Vol}(S^{m+1})/2 ? \) Pu Conj

**Example:** \( \text{FillVol}((S^m, d_{D^{m+1}}, [[S^m]])) \leq \text{Vol}(D^{m+1}) \).

**Pf:** Take \( N = D^{m+1}_+ \) so \( F : S^m \to \partial D^{m+1} \subset D^{m+1} \) is dist pres \( \square \)

**Thm:** If \( B(p, r) \) is a ball in an integral current space \( M \) then for a.e. \( r > 0 \) \( (B(p, r), d_M, [[B(p, r)]])) \) is an integral current space
Adapting Gromov’s Filling Volume [Portegies-Sormani]:

\[
\text{FillVol}(M^m) = \inf \{ M(N^{n+1}) | \partial N^{n+1} = M^m \}
\]

where the inf is over integral current spaces \( N^{n+1} = (X_N, d_N, T_N) \) such that \( \exists \) current preserving isometry \( F : M^m \to \partial N^{n+1} \).

Recall \( \partial N = (\text{set}(\partial T_N), d_N, \partial T_N) \) has the restricted distance \( d_N \) so \( d_N(F(p), F(q)) = d_M(p, q) \forall p, q \in X_M \) and \( F_{\#}\partial T_N = T_M \).

**Example:** \( \text{FillVol}((S^m, d_{S^m}, [[S^m]])) \leq \text{Vol}(S^{m+1})/2 \).

**Pf:** Take \( N = S^{m+1}_+ \) so \( F : S^m \to \text{Equator} \subset S^{m+1}_+ \) is dist pres \( \square \)

**Open:** Is \( \text{FillVol}((S^m, d_{S^m}, [[S^m]])) = \text{Vol}(S^{m+1})/2 \)? Pu Conj

**Example:** \( \text{FillVol}((S^m, d_{D^{m+1}}, [[S^m]])) \leq \text{Vol}(D^{m+1}) \).

**Pf:** Take \( N = D^{m+1}_+ \) so \( F : S^m \to \partial D^{m+1} \subset D^{m+1} \) is dist pres \( \square \)

**Thm:** If \( B(p, r) \) is a ball in an integral current space \( M \) then for a.e. \( r > 0 \) \( (B(p, r), d_M, [[B(p, r)]])) \) is an integral current space and so is \( \partial(B(p, r), d_M, [[B(p, r)]])) = (\partial B(p, r), d_M, [[\partial B(p, r)]]) \),
Adapting Gromov’s Filling Volume [Portegies-Sormani]:

\[ \text{FillVol}(M^m) = \inf \{ M(N^{n+1}) \mid \partial N^{n+1} = M^m \} \]

where the inf is over integral current spaces \( N^{n+1} = (X_N, d_N, T_N) \) such that \( \exists \) current preserving isometry \( F : M^m \to \partial N^{n+1} \).

Recall \( \partial N = (\text{set}(\partial T_N), d_N, \partial T_N) \) has the restricted distance \( d_N \) so \( d_N(F(p), F(q)) = d_M(p, q) \forall p, q \in X_M \) and \( F_\# \partial T_N = T_M \).

**Example:** \( \text{FillVol}((\mathbb{S}^m, d_{\mathbb{S}^m}, [[\mathbb{S}^m]]) \leq \text{Vol}(\mathbb{S}^{m+1})/2. \)

**Pf:** Take \( N = \mathbb{S}^{m+1}_+ \) so \( F : \mathbb{S}^m \to \text{Equator} \subset \mathbb{S}^{m+1}_+ \) is dist pres \( \square \)

**Open:** Is \( \text{FillVol}((\mathbb{S}^m, d_{\mathbb{S}^m}, [[\mathbb{S}^m]]) = \text{Vol}(\mathbb{S}^{m+1})/2? \) Pu Conj

**Example:** \( \text{FillVol}((\mathbb{S}^m, d_{\mathbb{D}^m}, [[\mathbb{S}^m]]) \leq \text{Vol}(\mathbb{D}^{m+1}). \)

**Pf:** Take \( N = \mathbb{D}^{m+1}_+ \) so \( F : \mathbb{S}^m \to \partial \mathbb{D}^{m+1} \subset \mathbb{D}^{m+1} \) is dist pres \( \square \)

**Thm:** If \( B(p, r) \) is a ball in an integral current space \( M \) then for a.e. \( r > 0 \) \( (B(p, r), d_M, [[B(p, r)]]) \) is an integral current space and so is \( \partial (B(p, r), d_M, [[B(p, r)]]) = (\partial B(p, r), d_M, [[\partial B(p, r)]]) \), and

\[ \text{FillVol}((\partial (B(p, r), d_M, [[B(p, r)]])) \leq M(B(p, r)). \]
Filling Volumes and Balls [Portegies-Sormani]:

\[ \text{FillVol}(M^m) = \inf \{ M(N^{n+1}) \mid \partial N^{n+1} = M^m \} \]

where the inf is over integral current spaces \( N^{n+1} = (X_N, d_N, T_N) \) such that \( \exists \) current preserving isometry \( F : M^m \to \partial N^{n+1} \).

Recall \( \partial N = (\text{set}(\partial T_N), d_N, \partial T_N) \) has the restricted distance \( d_N \) so \( d_N(F(p), F(q)) = d_M(p, q) \ \forall p, q \in X_M \) and \( F_\# \partial T_N = T_M \).

**Thm:** If \( B(p, r) \) is a ball in an integral current space \( M \) then for a.e. \( r > 0 \) \( (B(p, r), d_M, [[B(p, r)]])) \) is an integral current space and so is \( \partial(B(p, r), d_M, [[B(p, r)]])) = (\partial B(p, r), d_M, [[\partial B(p, r)]]), \) and

\[ \text{FillVol}(\partial(B(p, r), d_M, [[B(p, r)]])) \leq M(B(p, r)). \]
Filling Volumes and Balls [Portegies-Sormani]:

$$\text{FillVol}(M^m) = \inf \{ M(N^{n+1}) | \partial N^{n+1} = M^m \}$$

where the inf is over integral current spaces $N^{n+1} = (X_N, d_N, T_N)$ such that $\exists$ current preserving isometry $F : M^m \to \partial N^{n+1}$.

Recall $\partial N = (\text{set}(\partial T_N), d_N, \partial T_N)$ has the restricted distance $d_N$ so $d_N(F(p), F(q)) = d_M(p, q) \forall p, q \in X_M$ and $F_\# \partial T_N = T_M$.

**Thm:** If $B(p, r)$ is a ball in an integral current space $M$ then for a.e. $r > 0$ $(B(p, r), d_M, [[B(p, r)]]))$ is an integral current space and so is $\partial(B(p, r), d_M, [[B(p, r)]])) = (\partial B(p, r), d_M, [[\partial B(p, r)]]))$, and

$$\text{FillVol}(\partial(B(p, r), d_M, [[B(p, r)]])) \leq M(B(p, r)).$$

**Recall:** $p \in \text{set}(T)$ if $\lim \inf_{r \to 0} M(B(p, r))/r^m > 0$. 

Corollary: $p \in \text{set}(T)$ if $\lim \inf_{r \to 0} \text{FillVol}(\partial B(p, r))/r^m > 0$.

This corollary was applied by S-Wenger Matveev-Portegies to prove

**Thm:** $M_{GH} = M_{SWIF}$ for $M_j$ with $\text{Vol}(M_j) \geq V$ and $\text{Ricci} \geq H$.

**Pf:** Perelman Colding Gv: $\exists C_m H, V$ s.t. $\text{FillVol}(\partial B(p, r)) \geq C_m V, H r^m$.

combined with Corollary above and Portegies-Sormani (next slide) which says $B_j \text{SWIF} \rightarrow B_\infty = \Rightarrow \text{FillVol}(\partial B_j) \rightarrow \text{FillVol}(\partial B_\infty)$. $\square$
Filling Volumes and Balls [Portegies-Sormani]:

\[ \text{FillVol}(M^m) = \inf \{ M(N^{n+1}) | \partial N^{n+1} = M^m \} \]

where the inf is over integral current spaces \( N^{n+1} = (X_N, d_N, T_N) \) such that there exists a current preserving isometry \( F : M^m \to \partial N^{n+1} \).

Recall \( \partial N = (\text{set}(\partial T_N), d_N, \partial T_N) \) has the restricted distance \( d_N \) so \( d_N(F(p), F(q)) = d_M(p, q) \forall p, q \in X_M \) and \( F \# \partial T_N = T_M \).

**Thm:** If \( B(p, r) \) is a ball in an integral current space \( M \) then for a.e. \( r > 0 \) \( (B(p, r), d_M, [[B(p, r)]]) \) is an integral current space and so is \( \partial(B(p, r), d_M, [[B(p, r)]]) = (\partial B(p, r), d_M, [[\partial B(p, r)]]) \), and

\[ \text{FillVol}(\partial(B(p, r), d_M, [[B(p, r)]])) \leq M(B(p, r)). \]

**Recall:** \( p \in \text{set}(T) \) if \( \liminf_{r \to 0} M(B(p, r))/r^m > 0 \).

**Coro:** \( p \in \text{set}(T) \) if \( \liminf_{r \to 0} \text{FillVol}(\partial B(p, r))/r^m > 0 \).
Filling Volumes and Balls [Portegies-Sormani]:

\[
\text{FillVol}(M^m) = \inf \left\{ M(N^{n+1}) \mid \partial N^{n+1} = M^m \right\}
\]

where the inf is over integral current spaces \( N^{n+1} = (X_N, d_N, T_N) \) such that \( \exists \) current preserving isometry \( F : M^m \rightarrow \partial N^{n+1} \).

Recall \( \partial N = (\text{set}(\partial T_N), d_N, \partial T_N) \) has the restricted distance \( d_N \) so \( d_N(F(p), F(q)) = d_M(p, q) \ \forall \ p, q \in X_M \) and \( F\#\partial T_N = T_M \).

**Thm:** If \( B(p, r) \) is a ball in an integral current space \( M \) then for a.e. \( r > 0 \) \( (B(p, r), d_M, [[B(p, r)]] ) \) is an integral current space and so is \( \partial(B(p, r), d_M, [[B(p, r)]] ) = (\partial B(p, r), d_M, [[\partial B(p, r)]] ) \), and

\[
\text{FillVol}(\partial(B(p, r), d_M, [[B(p, r)]])) \leq M(B(p, r)).
\]

**Recall:** \( p \in \text{set}(T) \) if \( \lim \inf_{r \rightarrow 0} M(B(p, r))/r^m > 0 \).

**Coro:** \( p \in \text{set}(T) \) if \( \lim \inf_{r \rightarrow 0} \text{FillVol}(\partial B(p, r))/r^m > 0 \).

This corollary was applied by S-Wenger Matveev-Portegies to prove

**Thm:** \( M_{GH} = M_{SWIF} \) for \( M_j \) with \( \text{Vol}(M_j) \geq V \) and \( \text{Ricci} \geq H \).
Filling Volumes and Balls [Portegies-Sormani]:

\[
\text{FillVol}(M^m) = \inf \left\{ M(N^{n+1}) \mid \partial N^{n+1} = M^m \right\}
\]

where the inf is over integral current spaces \(N^{n+1} = (X_N, d_N, T_N)\) such that \(\exists\) current preserving isometry \(F : M^m \to \partial N^{n+1}\).

Recall \(\partial N = (\text{set}(\partial T_N), d_N, \partial T_N)\) has the restricted distance \(d_N\) so \(d_N(F(p), F(q)) = d_M(p, q) \forall p, q \in X_M\) and \(F_\# \partial T_N = T_M\).

**Thm:** If \(B(p, r)\) is a ball in an integral current space \(M\) then for a.e. \(r > 0\) \((B(p, r), d_M, [B(p, r)])\) is an integral current space and so is \(\partial(B(p, r), d_M, [B(p, r)]) = (\partial B(p, r), d_M, [\partial B(p, r)])\), and

\[
\text{FillVol}((\partial(B(p, r), d_M, [B(p, r)])) \leq M(B(p, r)).
\]

**Recall:** \(p \in \text{set}(T)\) if \(\liminf_{r \to 0} M(B(p, r))/r^m > 0\).

**Coro:** \(p \in \text{set}(T)\) if \(\liminf_{r \to 0} \text{FillVol}(\partial B(p, r))/r^m > 0\).

This corollary was applied by S-Wenger Matveev-Portegies to prove

**Thm:** \(M_{GH} = M_{SWIF}\) for \(M_j\) with \(\text{Vol}(M_j) \geq V\) and \(\text{Ricci} \geq H\).

**Pf:** Perelman Colding Gv: \(\exists C_{H,V}^m\) s.t. \(\text{FillVol}(\partial B(p, r)) \geq C_{V,H}^m r^m\).
Filling Volumes and Balls [Portegies-Sormani]:

\[
\text{FillVol}(M^m) = \inf \left\{ M(N^{n+1}) \mid \partial N^{n+1} = M^m \right\}
\]

where the inf is over integral current spaces \( N^{n+1} = (X_N, d_N, T_N) \) such that \( \exists \) current preserving isometry \( F : M^m \to \partial N^{n+1} \).

Recall \( \partial N = (\text{set}(\partial T_N), d_N, \partial T_N) \) has the restricted distance \( d_N \) so \( d_N(F(p), F(q)) = d_M(p, q) \forall p, q \in X_M \) and \( F_# \partial T_N = T_M \).

**Thm:** If \( B(p, r) \) is a ball in an integral current space \( M \) then for a.e. \( r > 0 \) \( (B(p, r), d_M, [[B(p, r)]])) \) is an integral current space and so is \( \partial(B(p, r), d_M, [[B(p, r)]])) = (\partial B(p, r), d_M, [[\partial B(p, r)]]), \) and

\[
\text{FillVol}((\partial(B(p, r), d_M, [[B(p, r)]]))) \leq M(B(p, r)).
\]

**Recall:** \( p \in \text{set}(T) \) if \( \lim \inf_{r \to 0} M(B(p, r))/r^m > 0 \).

**Coro:** \( p \in \text{set}(T) \) if \( \lim \inf_{r \to 0} \text{FillVol}(\partial B(p, r))/r^m > 0 \).

This corollary was applied by S-Wenger Matveev-Portegies to prove

**Thm:** \( M_{GH} = M_{SWIF} \) for \( M_j \) with \( \text{Vol}(M_j) \geq V \) and \( \text{Ricci} \geq H \).

**Pf:** Perelman Colding Gv: \( \exists C^m_{H, V} \) s.t. \( \text{FillVol}(\partial B(p, r)) \geq C^m_{V, H} r^m \).

combined with Corollary above and Portegies-Sormani (next slide) which says \( B_j \xrightarrow{\text{SWIF}} B_\infty \implies \text{FillVol}(\partial B_j) \to \text{FillVol}(\partial B_\infty). \) □
Filling Volumes and Balls [Portegies-Sormani]:

\[ \text{FillVol}(M^m) = \inf \{ M(N^{n+1}) \mid \partial N^{n+1} = M^m \} \]

where the inf is over integral current spaces \( N^{n+1} = (X_N, d_N, T_N) \) such that \( \exists \) current preserving isometry \( F : M^m \to \partial N^{n+1} \).

Recall \( \partial N = (\text{set}(\partial T_N), d_N, \partial T_N) \) has the restricted distance \( d_N \) so \( d_N(F(p), F(q)) = d_M(p, q) \ \forall p, q \in X_M \) and \( F \# \partial T_N = T_M \).

**Thm:** If \( B(p, r) \) is a ball in an integral current space \( M \) then for a.e. \( r > 0 \) \( (B(p, r), d_M, [[B(p, r)]])) \) is an integral current space and so is \( \partial(B(p, r), d_M, [[B(p, r)]])) = (\partial B(p, r), d_M, [[\partial B(p, r)]])), and

\[ \text{FillVol}(\partial(B(p, r), d_M, [[B(p, r)]])) \leq M(B(p, r)). \]

Recall: \( p \in \text{set}(T) \) if \( \lim \inf_{r \to 0} M(B(p, r))/r^m > 0 \).

**Coro:** \( p \in \text{set}(T) \) if \( \lim \inf_{r \to 0} \text{FillVol}(\partial B(p, r))/r^m > 0 \).

This corollary was applied by S-Wenger Matveev-Portegies to prove

**Thm:** \( M_{GH} = M_{SWIF} \) for \( M_j \) with \( \text{Vol}(M_j) \geq V \) and \( \text{Ricci} \geq H \).

**Pf:** Perelman Colding Gv: \( \exists C_{H, V}^m \) s.t. \( \text{FillVol}(\partial B(p, r)) \geq C_{V, H}^m r^m \). combined with Corollary above and Portegies-Sormani (next slide) which says \( B_j \xrightarrow{\text{SWIF}} B_\infty \implies \text{FillVol}(\partial B_j) \to \text{FillVol}(\partial B_\infty). \) \( \square \)
Filling Volumes and Balls [Portegies-Sormani]:

\[ \text{FillVol}(M^m) = \inf \left\{ M(N^{n+1}) \mid \partial N^{n+1} = M^m \right\} \]

where the inf is over integral current spaces \( N^{n+1} = (X_N, d_N, T_N) \) such that \( \exists \) current preserving isometry \( F : M^m \to \partial N^{n+1} \).

Recall \( \partial N = (\text{set}(\partial T_N), d_N, \partial T_N) \) has the restricted distance \( d_N \) so \( d_N(F(p), F(q)) = d_M(p, q) \forall p, q \in X_M \) and \( F_# \partial T_N = T_M \).

**Thm:** If \( B(p, r) \) is a ball in an integral current space \( M \) then for a.e. \( r > 0 \) \( (B(p, r), d_M, [[B(p, r)]])) \) is an integral current space and so is \( \partial(B(p, r), d_M, [[B(p, r)]])) = (\partial B(p, r), d_M, [[\partial B(p, r)]]), \) and

\[ \text{FillVol}((\partial(B(p, r), d_M, [[B(p, r)]])) \leq M(B(p, r)). \]

**Recall:** \( p \in \text{set}(T) \) if \( \lim \inf_{r \to 0} M(B(p, r))/r^m > 0. \)

**Coro:** \( p \in \text{set}(T) \) if \( \lim \inf_{r \to 0} \text{FillVol}(\partial B(p, r))/r^m > 0. \)

This corollary was applied by S-Wenger Matveev-Portegies to prove

**Thm:** \( M_{GH} = M_{SWIF} \) for \( M_j \) with \( \text{Vol}(M_j) \geq V \) and \( \text{Ricci} \geq H. \)

**Pf:** Perelman Colding Gv: \( \exists C^m_{H, V} \text{ s.t. } \text{FillVol}(\partial B(p, r)) \geq C^m_{V, H} r^m. \)

combined with Corollary above and Portegies-Sormani (next slide) which says \( B_j \xrightarrow{\text{SWIF}} B_\infty \implies \text{FillVol}(\partial B_j) \to \text{FillVol}(\partial B_\infty). \) □
Filling Volumes and SWIF Limits [Portegies-Sormani]:

Thm: $M_j^m \xrightarrow{\text{SWIF}} M_\infty \implies \text{FillVol}(\partial M_j^m) \to \text{FillVol}(\partial M_\infty)$.

Proof:

We need only show that for any fixed $\epsilon > 0$\n\[
\text{FillVol}(\partial M_j^m) \leq d_{\text{SWIF}}(M_j^m, M_j^m) + \text{FillVol}(\partial M_j^m) + \epsilon.
\]

1. By defn:\n\[\exists \varphi_i: X_i \to Z \text{ and } A + \partial B = \varphi_1#T_1 - \varphi_2#T_2 \text{ s.t. } M(A) + M(B) \leq d_{\text{SWIF}}(M_j^m, M_j^m) + \epsilon/2.\]

2. $\partial A = \partial \varphi_1#T_1 - \partial \varphi_2#T_2 - \partial \partial B$.

Let $N_j^m = (\text{set}(A), d_Z, A)$ so $\varphi_i: \partial M_i \to \partial N_j^m \subset Z$.

3. By defn of FillVol:\n\[\exists N_j^m_2 with \partial N_j^m_2 = \partial M_j^m_2 \text{ such that } M(N_j^m_2) \leq \text{FillVol}(\partial M_j^m) + \epsilon/2.\]

4. Glue $N_j^m_1$ to $N_j^m_2$ along $\partial M_j^m_2$ to obtain $N_j^m_1, 2$ s.t. $\partial N_j^m_1, 2 = \partial M_j^m_1$.

\[M(N_j^m_1, 2) \leq M(N_j^m_1) + M(N_j^m_2) \leq d_{\text{SWIF}}(M_j^m, M_j^m) + \text{FillVol}(\partial M_j^m) + \epsilon.\]
Filling Volumes and SWIF Limits [Portegies-Sormani]:

**Thm:** \( M_j^m \xrightarrow{\text{SWIF}} M_\infty^m \implies \text{FillVol}(\partial M_j^m) \to \text{FillVol}(\partial M_\infty^m). \)

**Proof:** We need only show that for any fixed \( \epsilon > 0 \)

\[
\text{FillVol}(\partial M_1^m) \leq d_{\text{SWIF}}(M_1^m, M_2^m) + \text{FillVol}(\partial M_2^m) + \epsilon.
\]
Filling Volumes and SWIF Limits [Portegies-Sormani]:

**Thm:** \( M^m_j \xrightarrow{\text{SWIF}} M^m_\infty \implies \text{FillVol}(\partial M^m_j) \to \text{FillVol}(\partial M^m_\infty) \).

**Proof:** We need only show that for any fixed \( \epsilon > 0 \)

\[
\text{FillVol}(\partial M^m_1) \leq d_{\text{SWIF}}(M^m_1, M^m_2) + \text{FillVol}(\partial M^m_2) + \epsilon.
\]
Filling Volumes and SWIF Limits [Portegies-Sormani]:

**Thm:** \( M^m_j \xrightarrow{\text{SWIF}} M^m_\infty \implies \text{FillVol}(\partial M^m_j) \to \text{FillVol}(\partial M^m_\infty) \).

**Proof:** We need only show that for any fixed \( \epsilon > 0 \)

\[
\text{FillVol}(\partial M^m_1) \leq d_{\text{SWIF}}(M^m_1, M^m_2) + \text{FillVol}(\partial M^m_2) + \epsilon.
\]

1. \[ \textbf{M}_1 \quad \textbf{B} \quad \textbf{M}_2 \]
2. \[ \partial \textbf{M}_1 \quad \textbf{N}_1 \quad \partial \textbf{M}_2 \]
3. \[ \partial \textbf{M}_2 \quad \textbf{N}_2 \]
4. \[ \partial \textbf{M}_1 \quad \textbf{N}_{1,2} \]
Filling Volumes and SWIF Limits [Portegies-Sormani]:

**Thm:** \( M^m_j \xrightarrow{\text{SWIF}} M^m_\infty \implies \text{FillVol}(\partial M^m_j) \to \text{FillVol}(\partial M^m_\infty). \)

**Proof:** We need only show that for any fixed \( \epsilon > 0 \)
\[
\text{FillVol}(\partial M^m_1) \leq d_{\text{SWIF}}(M^m_1, M^m_2) + \text{FillVol}(\partial M^m_2) + \epsilon.
\]

1. By defn: \( \exists \varphi_i : X_i \to Z \) and \( A + \partial B = \varphi_1\# T_1 - \varphi_2\# T_2 \) s.t. \( M(A) + M(B) \leq d_{\text{SWIF}}(M^m_1, M^m_2) + \epsilon/2. \)
Filling Volumes and SWIF Limits [Portegies-Sormani]:

Thm: \[ M_j^m \xrightarrow{\text{SWIF}} M_\infty^m \implies \text{FillVol}(\partial M_j^m) \to \text{FillVol}(\partial M_\infty^m). \]

Proof: We need only show that for any fixed \( \epsilon > 0 \)

\[
\text{FillVol}(\partial M_1^m) \leq d_{\text{SWIF}}(M_1^m, M_2^m) + \text{FillVol}(\partial M_2^m) + \epsilon.
\]

1. By defn: \( \exists \varphi_i : X_i \to Z \) and \( A + \partial B = \varphi_1\# T_1 - \varphi_2\# T_2 \) s.t. \( M(A) + M(B) \leq d_{\text{SWIF}}(M_1^m, M_2^m) + \epsilon/2. \)

2. \( \partial A = \partial \varphi_1\# T_1 - \partial \varphi_2\# T_2 - \partial \partial B \)
Filling Volumes and SWIF Limits [Portegies-Sormani]:

**Thm:** \(M_j^m \xrightarrow{\text{SWIF}} M_\infty \Rightarrow \text{FillVol}(\partial M_j^m) \to \text{FillVol}(\partial M_\infty).\)

**Proof:** We need only show that for any fixed \(\epsilon > 0\)

\[
\text{FillVol}(\partial M_1^m) \leq d_{\text{SWIF}}(M_1^m, M_2^m) + \text{FillVol}(\partial M_2^m) + \epsilon.
\]

1. By defn: \(\exists \varphi_i : X_i \to Z\) and \(A + \partial B = \varphi_1\# T_1 - \varphi_2\# T_2\) s.t. \(M(A) + M(B) \leq d_{\text{SWIF}}(M_1^m, M_2^m) + \epsilon/2.\)
2. \(\partial A = \partial \varphi_1\# T_1 - \partial \varphi_2\# T_2 - \partial \partial B = \varphi_1\# \partial T_1 - \varphi_2\# \partial T_2.\)
Filling Volumes and SWIF Limits [Portegies-Sormani]:

**Thm:** \( M^m_j \xrightarrow{\text{SWIF}} M^m_\infty \implies \text{FillVol}(\partial M^m_j) \to \text{FillVol}(\partial M^m_\infty). \)

**Proof:** We need only show that for any fixed \( \epsilon > 0 \)

\[
\text{FillVol}(\partial M^m_1) \leq d_{\text{SWIF}}(M^m_1, M^m_2) + \text{FillVol}(\partial M^m_2) + \epsilon.
\]

1. By defn: \( \exists \varphi_i : X_i \to Z \) and \( A + \partial B = \varphi_1 \# T_1 - \varphi_2 \# T_2 \) s.t.

\[
M(A) + M(B) \leq d_{\text{SWIF}}(M^m_1, M^m_2) + \epsilon/2.
\]

2. \( \partial A = \partial \varphi_1 \# T_1 - \partial \varphi_2 \# T_2 - \partial \partial B = \varphi_1 \# \partial T_1 - \varphi_2 \# \partial T_2. \)

Let \( N^m_i = (\text{set}(A), d_Z, A) \) so \( \varphi_i : \partial M_i \to \partial N_1 \subset Z. \)
Filling Volumes and SWIF Limits [Portegies-Sormani]:

**Thm:** \(M_j^m \xrightarrow{\text{SWIF}} M_\infty^m \implies \text{FillVol}(\partial M_j^m) \to \text{FillVol}(\partial M_\infty^m).\)

**Proof:** We need only show that for any fixed \(\epsilon > 0\)

\[
\text{FillVol}(\partial M_1^m) \leq d_{\text{SWIF}}(M_1^m, M_2^m) + \text{FillVol}(\partial M_2^m) + \epsilon.
\]

1. By defn: \(\exists \varphi_i : X_i \to Z\) and \(A + \partial B = \varphi_1# T_1 - \varphi_2# T_2\) s.t.
   \(M(A) + M(B) \leq d_{\text{SWIF}}(M_1^m, M_2^m) + \epsilon/2.\)

2. \(\partial A = \partial \varphi_1# T_1 - \partial \varphi_2# T_2 - \partial \partial B = \varphi_1\partial T_1 - \varphi_2\partial T_2.\)
   
   Let \(N_1^m = (\text{set}(A), d_Z, A)\) so \(\varphi_i : \partial M_i \to \partial N_1 \subset Z.\)

3. By defn of FillVol: \(\exists N_2^m\) with \(\partial N_2^m = \partial M_2^m\) such that
   \(M(N_2^m) \leq \text{FillVol}(\partial M_2^m) + \epsilon/2.\)
Filling Volumes and SWIF Limits [Portegies-Sormani]:

Thm: \(M_j^m \xrightarrow{\text{SWIF}} M_\infty^m \implies \text{FillVol}(\partial M_j^m) \rightarrow \text{FillVol}(\partial M_\infty^m)\).

Proof: We need only show that for any fixed \(\epsilon > 0\)

\[
\text{FillVol}(\partial M_1^m) \leq d_{SWIF}(M_1^m, M_2^m) + \text{FillVol}(\partial M_2^m) + \epsilon.
\]

1. By defn: \(\exists \varphi_i : X_i \rightarrow Z\) and \(A + \partial B = \varphi_1\# T_1 - \varphi_2\# T_2\) s.t. \(M(A) + M(B) \leq d_{SWIF}(M_1^m, M_2^m) + \epsilon/2\).

2. \(\partial A = \partial \varphi_1\# T_1 - \partial \varphi_2\# T_2 - \partial \partial B = \varphi_1\# \partial T_1 - \varphi_2\# \partial T_2\).

   Let \(N_1^m = (\text{set}(A), d_Z, A)\) so \(\varphi_i : \partial M_i \rightarrow \partial N_1 \subset Z\).

3. By defn of FillVol: \(\exists N_2^m\) with \(\partial N_2^m = \partial M_2^m\) such that

   \[
   M(N_2^m) \leq \text{FillVol}(\partial M_2^m) + \epsilon/2.
   \]

4. Glue \(N_1^m\) to \(N_2^m\) along \(\partial M_2^m\) to obtain \(N_{1,2}^m\) s.t. \(\partial N_{1,2}^m = \partial M_1^m\).
Filling Volumes and SWIF Limits [Portegies-Sormani]:

**Thm:** $M_j^m \xrightarrow{\text{SWIF}} M^m \implies \text{FillVol}(\partial M_j^m) \to \text{FillVol}(\partial M^m)$.

**Proof:** We need only show that for any fixed $\epsilon > 0$

$$\text{FillVol}(\partial M^m_1) \leq d_{\text{SWIF}}(M^m_1, M^m_2) + \text{FillVol}(\partial M^m_2) + \epsilon.$$

1. By defn: $\exists \varphi_i : X_i \to Z$ and $A + \partial B = \varphi_1#T_1 - \varphi_2#T_2$ s.t. $M(A) + M(B) \leq d_{\text{SWIF}}(M^m_1, M^m_2) + \epsilon/2$.

2. $\partial A = \partial \varphi_1#T_1 - \partial \varphi_2#T_2 - \partial \partial B = \varphi_1#\partial T_1 - \varphi_2#\partial T_2$.

Let $N^m_1 = (\text{set}(A), d_Z, A)$ so $\varphi_i : \partial M_i \to \partial N_1 \subset Z$.

3. By defn of FillVol: $\exists N^m_2$ with $\partial N^m_2 = \partial M^m_2$ such that $M(N^m_2) \leq \text{FillVol}(\partial M^m_2) + \epsilon/2$.

4. Glue $N^m_1$ to $N^m_2$ along $\partial M^m_2$ to obtain $N^m_{1,2}$ s.t. $\partial N^m_{1,2} = \partial M^m_1$

$M(N^m_{1,2}) \leq M(N^m_1) + M(N^m_2) \leq$
Filling Volumes and SWIF Limits [Portegies-Sormani]:

**Thm:** $M_j^m \xrightarrow{\text{SWIF}} M_\infty^m \implies \text{FillVol}(\partial M_j^m) \to \text{FillVol}(\partial M_\infty^m)$.

**Proof:** We need only show that for any fixed $\epsilon > 0$

$$\text{FillVol}(\partial M_1^m) \leq d_{\text{SWIF}}(M_1^m, M_2^m) + \text{FillVol}(\partial M_2^m) + \epsilon.$$

1. By defn: $\exists \varphi_i : X_i \to Z$ and $A + \partial B = \varphi_1 T_1 - \varphi_2 T_2$ s.t. $M(A) + M(B) \leq d_{\text{SWIF}}(M_1^m, M_2^m) + \epsilon/2$.

2. $\partial A = \partial \varphi_1 T_1 - \partial \varphi_2 T_2 - \partial \partial B = \varphi_1 \partial T_1 - \varphi_2 \partial T_2$.

Let $N_1^m = (\text{set}(A), d_Z, A)$ so $\varphi_i : \partial M_i \to \partial N_1 \subset Z$.

3. By defn of FillVol: $\exists N_2^m$ with $\partial N_2^m = \partial M_2^m$ such that

$$M(N_2^m) \leq \text{FillVol}(\partial M_2^m) + \epsilon/2.$$

4. Glue $N_1^m$ to $N_2^m$ along $\partial M_2^m$ to obtain $N_{1,2}^m$ s.t. $\partial N_{1,2}^m = \partial M_1^m$

$$M(N_{1,2}) \leq M(N_1^m) + M(N_2^m) \leq d_{\text{SWIF}}(M_1, M_2) + \text{FillVol}(\partial M_2^m) + \epsilon.$$
Intrinsic Flat and Gromov-Hausdorff Convergence

Christina Sormani

CUNY GC and Lehman College

Lectures IV: Proving Intrinsic Flat Convergence
Volume Preserving Intrinsic Flat $\mathcal{VF}$ Convergence

Defn: $M_j \xrightarrow{\mathcal{VF}} M_\infty$ if $d_F(M_j, M_\infty) = d_{SWIF}(M_j, M_\infty) \to 0$.

Defn: $M_j \xrightarrow{F} M_\infty$ if $\text{Vol}(M_j) \to \text{Vol}(M_\infty)$. 
Volume Preserving Intrinsic Flat $\mathcal{VF}$ Convergence

Defn: $M_j \xrightarrow{\mathcal{VF}} M_\infty$ if $M_j \xrightarrow{\mathcal{F}} M_\infty$ and $\text{Vol}(M_j) \to \text{Vol}(M_\infty)$. 
Volume Preserving Intrinsic Flat $\mathcal{VF}$ Convergence

Defn: $M_j \xrightarrow{\mathcal{VF}} M_\infty$ if $M_j \xrightarrow{\mathcal{F}} M_\infty$ and $\text{Vol}(M_j) \to \text{Vol}(M_\infty)$.

Defn: $M_j \xrightarrow{\mathcal{F}} M_\infty$ if $d_{\mathcal{F}}(M_j, M_\infty) = d_{\text{SWIF}}(M_j, M_\infty) \to 0$:

Sormani-Wenger: Intrinsic Flat Distance

The intrinsic flat distance between oriented manifolds $M_i^m$ is:

$$d_{\text{SWIF}}(M_1^m, M_2^m) = \inf \left\{ d_F^Z (\varphi_1\#[[M_1^m]], \varphi_2\#[[M_2^m]]) \mid \varphi_i : M_i^m \to Z \right\}$$

where the infimum is taken over all complete metric spaces, $Z$, and over all distance preserving maps $\varphi_i : M_i^m \to Z$.

Here: $d_F^Z (\varphi_1\#[[M_1^m]], \varphi_2\#[[M_2^m]])$ is the Federer-Fleming Flat dist

$$= \inf \left\{ \text{M}_{\text{area}}(A) + \text{M}_{\text{vol}}(B) : A + \partial B = \varphi_1\#[[M_1^m]] - \varphi_2\#[[M_2^m]] \right\}$$
Suppose \((M_1, g_1)\) and \((M_2, g_2)\) are oriented precompact Riemannian manifolds with diffeomorphic subregions \(W_i \subset M_i\). Identifying \(W_1 = W_2 = W\) assume that on \(W\) we have

\[g_1 \leq (1 + \varepsilon)^2 g_2 \text{ and } g_2 \leq (1 + \varepsilon)^2 g_1.\]

Taking the extrinsic diameters,

\[\text{diam}(M_i) \leq D\]

we define a hemispherical width,

\[a > \frac{\arccos(1 + \varepsilon)^{-1}}{\pi} D.\]

to find dist pres maps

\[g \text{ on } \omega \times [0, a]\]

is

\[g = dr^2 + f^2(r) g_i;\]

\[f(r) = \max \{ \cos(r), \cos(a-r) \}^3\]
Lakzian-Sormani: Estimating $d_{SWIF}$

Lakzian-Sormani: Suppose $(M_1, g_1)$ and $(M_2, g_2)$ are oriented precompact Riemannian manifolds with diffeomorphic subregions $W_i \subset M_i$. Identifying $W_1 = W_2 = W$ assume that on $W$ we have

$$g_1 \leq (1 + \varepsilon)^2 g_2 \text{ and } g_2 \leq (1 + \varepsilon)^2 g_1.$$  

Taking the extrinsic diameters,

$$\text{diam}(M_i) \leq D$$  

we define a hemispherical width,

$$a > \frac{\arccos(1 + \varepsilon)^{-1}}{\pi} D.$$  

Taking the difference in distances with respect to the outside manifolds, we set

$$\lambda = \sup_{x,y \in W} |d_{M_1}(x, y) - d_{M_2}(x, y)| \leq 2D,$$

and we define the height,

$$\overline{h} = \max\{ \sqrt{2\lambda D}, D \sqrt{\varepsilon^2 + 2\varepsilon} \}.$$  

Then taking $Z = M_1 \sqcup W_1 \times [0, h] \sqcup \omega \times [0, \sigma] \sqcup \omega_2 \times [0, h] \sqcup M_2$

$$d_{\varphi}(M_1, M_2) \leq (2\overline{h} + a) \left( \text{Vol}_m(W_1) + \text{Vol}_m(W_2) + \text{Vol}_{m-1}(\partial W_1) + \text{Vol}_{m-1}(\partial W_2) \right) + \text{Vol}_m(M_1 \setminus W_1) + \text{Vol}_m(M_2 \setminus W_2),$$
Lakzian-Sormani: Estimating $d_{SWIF}$

Lakzian-Sormani: Suppose $(M_1, g_1)$ and $(M_2, g_2)$ are oriented precompact Riemannian manifolds with diffeomorphic subregions $W_i \subset M_i$. Identifying $W_1 = W_2 = W$ assume that on $W$ we have

$$g_1 \leq (1 + \varepsilon)^2 g_2 \text{ and } g_2 \leq (1 + \varepsilon)^2 g_1.$$ 

Taking the extrinsic diameters,

$$\text{diam}(M_i) \leq D$$

we define a hemispherical width,

$$a > \frac{\arccos(1 + \varepsilon)^{-1}}{\pi} D.$$ 

Taking the difference in distances with respect to the outside manifolds, we set

$$\lambda = \sup_{x,y \in W} |d_M(x,y) - d_{M_2}(x,y)| \leq 2D,$$

and we define the height,

$$\overline{h} = \max\{\sqrt{2\lambda}D, D \sqrt{\varepsilon^2 + 2\varepsilon}\}.$$ 

Then taking $Z = M_1 \cup W_1 \cup \partial W_1 \cup \partial W_2$, we have

$$d_{\mathcal{S}}(M_1, M_2) \leq (2\overline{h} + a) \left( \text{Vol}_m(W_1) + \text{Vol}_m(W_2) + \text{Vol}_{m-1}(\partial W_1) + \text{Vol}_{m-1}(\partial W_2) \right)$$

$$+ \text{Vol}_m(M_1 \setminus W_1) + \text{Vol}_m(M_2 \setminus W_2).$$
Defn: Volume Above Distance Below Conv: $M_j \xrightarrow{\text{VADB}} M_\infty \implies M_j \xrightarrow{\mathcal{V}_F} M_\infty$.

If $Vol_j(M_j) \to Vol_\infty(M_\infty)$ and $\exists D > 0$ s.t. $\text{Diam}(M_j) \leq D$ and $\exists C^1$ diffeomorphism $\psi_j : M_\infty \to M_j$ such that

$$d_j(\psi_j(p), \psi_j(q)) \geq d_\infty(p, q) \ \forall p, q \in M_\infty.$$
Allen-Perales-Sormani VADB

**Allen-Perales-Sormani:** [arXiv:2003.01172]

\[ M_j \xrightarrow{\text{VADB}} M_{\infty} \implies M_j \xrightarrow{\nu F} M_{\infty}. \]

**Defn:** *Volume Above Distance Below Conv:* \( M_j \xrightarrow{\text{VADB}} M_{\infty} \) if \( \text{Vol}_j(M_j) \to \text{Vol}_\infty(M_{\infty}) \) and \( \exists D > 0 \) s.t. \( \text{Diam}(M_j) \leq D \) and \( \exists C^1 \) diffeomorphism \( \psi_j : M_{\infty} \to M_j \) such that

\[ d_j(\psi_j(p), \psi_j(q)) \geq d_\infty(p, q) \quad \forall p, q \in M_{\infty}. \]

An earlier theorem that inspired us:

**Huang-Lee-Sormani:** Given \((M, d_0)\) Riemannian without boundary and fix \(\lambda > 0\), suppose that \(d_j\) are length metrics on \(M\) such that

\[ \lambda \geq \frac{d_j(p, q)}{d_0(p, q)} \geq \frac{1}{\lambda}. \]

Then there exists a subsequence, also denoted \(d_j\), and a length metric \(d_\infty\) such that \(d_j\) converges uniformly to \(d_\infty\):

\[ \varepsilon_j = \sup \{|d_j(p, q) - d_\infty(p, q)| : p, q \in X\} \to 0. \]

and \(M_j\) converges in the intrinsic flat and Gromov-Hausdorff sense to \(M_{\infty}\):

\[ M_j \xrightarrow{F} M_{\infty} \quad \text{and} \quad M_j \xrightarrow{GH} M_{\infty} \]

where \(M_j = (M, d_j)\) and \(M_{\infty} = (M, d_\infty)\).
Let $M$ be an oriented, connected and closed manifold, $M_j = (M, g_j)$ and $M_0 = (M, g_0)$ be Riemannian manifolds with $\text{Diam}(M_j) \leq D$, $\text{Vol}_j(M_j) \leq V$ and $F_j : M_j \to M_0$ a $C^1$ diffeomorphism and distance non-increasing map:

\[ d_j(x, y) \geq d_0(F_j(x), F_j(y)) \quad \forall x, y \in M_j. \]

Let $W_j \subset M_j$ be a measurable set and assume that there exists a $\delta_j > 0$ so that

\[ d_j(x, y) \leq d_0(F_j(x), F_j(y)) + 2\delta_j \quad \forall x, y \in W_j \]

with

\[ \text{Vol}_j(M_j \setminus W_j) \leq V_j \]

and

\[ h_j \geq \sqrt{2\delta_j D + \delta_j^2} \]

then

\[ d_F(M_0, M_j) \leq 2V_j + h_j V. \]
Let $M$ be an oriented, connected and closed manifold, $M_j = (M, g_j)$ and $M_0 = (M, g_0)$ be Riemannian manifolds with $\text{Diam}(M_j) \leq D$, $\text{Vol}_j(M_j) \leq V$ and $F_j : M_j \to M_0$ a $C^1$ diffeomorphism and distance non-increasing map:

\[(120) \quad d_j(x, y) \geq d_0(F_j(x), F_j(y)) \quad \forall x, y \in M_j.\]

Let $W_j \subset M_j$ be a measurable set and assume that there exists a $\delta_j > 0$ so that

\[(121) \quad d_j(x, y) \leq d_0(F_j(x), F_j(y)) + 2\delta_j \quad \forall x, y \in W_j\]

with

\[(122) \quad \text{Vol}_j(M_j \setminus W_j) \leq V_j\]

and

\[(123) \quad h_j \geq \sqrt{2\delta_j D + \delta_j^2}\]

then

\[(124) \quad d_F(M_0, M_j) \leq 2V_j + h_j V.\]
Let $M$ be an oriented, connected and closed manifold, $M_j = (M, g_j)$ and $M_0 = (M, g_0)$ be Riemannian manifolds with $\text{Diam}(M_j) \leq D$, $\text{Vol}_j(M_j) \leq V$ and $F_j : M_j \to M_0$ a $C^1$ diffeomorphism and distance non-increasing map:

\begin{equation}
\forall x, y \in M_j.
\end{equation}

Let $W_j \subset M_j$ be a measurable set and assume that there exists a $\delta_j > 0$ so that

\begin{equation}
\forall x, y \in W_j
\end{equation}

with

\begin{equation}
\text{Vol}_j(M_j \setminus W_j) \leq V_j
\end{equation}

and

\begin{equation}
h_j \geq \sqrt{2\delta_j D + \delta_j^2}
\end{equation}

then

\begin{equation}
d_F(M_0, M_j) \leq 2V_j + h_j V.
\end{equation}

$Z$ is $M_j$ glued along $W_j$ to $M_j \times [0, h]$ glued along $F_j(W_j)$ to $M_0$. 

\[ Z = \text{M}_j \times [0, h] \]
Allen-Sormani VADB to ptwise a.e. on $M \times M$

**Allen-Sormani:** If $(M, g_j)$ are compact continuous Riemannian manifolds without boundary and $(M, g_0)$ is a smooth Riemannian manifold such that

\[ g_j(v, v) \geq g_0(v, v) \quad \forall v \in T_p M \]

and

\[ \text{Vol}_j(M) \to \text{Vol}_0(M) \]

then there exists a subsequence such that

\[ \lim_{j \to \infty} d_j(p, q) = d_0(p, q) \] pointwise a.e. $(p, q) \in M \times M$. 

**Allen-Sormani VADB to ptwise a.e. on \( M \times M \)**

**Allen-Sormani:** If \((M,g_j)\) are compact continuous Riemannian manifolds without boundary and \((M,g_0)\) is a smooth Riemannian manifold such that

\[(85)\]
\[g_j(v,v) \geq g_0(v,v) \quad \forall v \in T_p M\]

and

\[(86)\]
\[\text{Vol}_j(M) \to \text{Vol}_0(M)\]

then there exists a subsequence such that

\[(87)\]
\[\lim_{j \to \infty} d_j(p,q) = d_0(p,q) \text{ pointwise a.e. } (p,q) \in M \times M.\]

**Figure 2.** A tube \( \mathcal{T} \) foliated by \( g_0 \)-geodesics, \( \gamma \), with \( L_j(\gamma) \geq L_0(\gamma) \) has \( \text{Vol}_j(\mathcal{T}) \to \text{Vol}_0(\mathcal{T}) \) so \( L_j(\gamma) \to L_0(\gamma) \) for almost every \( \gamma \) but not for \( \gamma \) ending at a tip.
Allen-Sormani VADB to ptwise a.e. on $M \times M$

**Allen-Sormani**: If $(M, g_j)$ are compact continuous Riemannian manifolds without boundary and $(M, g_0)$ is a smooth Riemannian manifold such that
\[ g_j(v, v) \geq g_0(v, v) \quad \forall v \in T_p M \]
and
\[ \operatorname{Vol}_j(M) \to \operatorname{Vol}_0(M) \]
then there exists a subsequence such that
\[ \lim_{j \to \infty} d_j(p, q) = d_0(p, q) \text{ pointwise a.e. } (p, q) \in M \times M. \]

**Figure 2.** A tube $\mathcal{T}$ foliated by $g_0$-geodesics, $\gamma$, with $L_j(\gamma) \geq L_0(\gamma)$ has $\operatorname{Vol}_j(\mathcal{T}) \to \operatorname{Vol}_0(\mathcal{T})$ so $L_j(\gamma) \to L_0(\gamma)$ for almost every $\gamma$ but not for $\gamma$ ending at a tip.

How to find a $W \subset M$ controlling $d(p, q)$ for all $p, q \in W$?
Allen-Sormani VADB to ptwise a.e. on $M \times M$

**Allen-Sormani:** If $(M, g_j)$ are compact continuous Riemannian manifolds without boundary and $(M, g_0)$ is a smooth Riemannian manifold such that

\[(85) \quad g_j(v, v) \geq g_0(v, v) \quad \forall v \in T_pM\]

and

\[(86) \quad \text{Vol}_j(M) \to \text{Vol}_0(M)\]

then there exists a subsequence such that

\[(87) \quad \lim_{j \to \infty} d_j(p, q) = d_0(p, q) \text{ pointwise a.e. } (p, q) \in M \times M.\]

**Figure 2.** A tube $T$ foliated by $g_0$-geodesics, $\gamma$, with $L_j(\gamma) \geq L_0(\gamma)$ has $\text{Vol}_j(T) \to \text{Vol}_0(T)$ so $L_j(\gamma) \to L_0(\gamma)$ for almost every $\gamma$ but not for $\gamma$ ending at a tip.

How to find a $W \subset M$ controlling $d(p, q)$ for all $p, q \in W$? Egoroff’s Theorem?
Allen-Sormani VADB to ptwise a.e. on $M \times M$

**Allen-Sormani:** If $(M, g_j)$ are compact continuous Riemannian manifolds without boundary and $(M, g_0)$ is a smooth Riemannian manifold such that

\begin{equation}
  g_j(v, v) \geq g_0(v, v) \quad \forall v \in T_p M
\end{equation}

and

\begin{equation}
  \text{Vol}_j(M) \to \text{Vol}_0(M)
\end{equation}

then there exists a subsequence such that

\begin{equation}
  \lim_{j \to \infty} d_j(p, q) = d_0(p, q) \text{ pointwise a.e. } (p, q) \in M \times M.
\end{equation}

**Figure 2.** A tube $\mathcal{T}$ foliated by $g_0$-geodesics, $\gamma$, with $L_j(\gamma) \geq L_0(\gamma)$ has $\text{Vol}_j(\mathcal{T}) \to \text{Vol}_0(\mathcal{T})$ so $L_j(\gamma) \to L_0(\gamma)$ for almost every $\gamma$ but not for $\gamma$ ending at a tip.

How to find a $W \subset M$ controlling $d(p, q)$ for all $p, q \in W$? Egoroff’s Theorem? But Egoroff’s Theorem only gives a set $S \in M \times M$ controlling $d(p, q)$ uniformly $\forall (p, q) \in S$...
Now we apply Egoroff’s theorem to obtain uniform convergence on a set of almost full measure.

**Proposition 7**. Under the hypotheses of Theorem 4.1, for every $\varepsilon > 0$ there exists a $d\text{vol}_{\mathcal{g}_0} \times d\text{vol}_{\mathcal{g}_0}$ measurable set, $S_{\varepsilon} \subset M \times M$, such that

\[
\sup\{|d_j(p,q) - d_0(p,q)| : (p,q) \in S_{\varepsilon}\} = \delta_{\varepsilon,j} \to 0,
\]

\[
\text{Vol}_{0 \times 0}(S_{\varepsilon}) > (1 - \varepsilon) \text{Vol}_{0 \times 0}(M \times M).
\]

and

\[
(p,q) \in S_{\varepsilon} \iff (q,p) \in S_{\varepsilon}.
\]
Now we apply Egoroff’s theorem to obtain uniform convergence on a set of almost full measure.

**Proposition.** Under the hypotheses of Theorem 4.1, for every $\varepsilon > 0$ there exists a $dvol_{g_0} \times dvol_{g_0}$ measurable set, $S_\varepsilon \subset M \times M$, such that

$$
\sup \{ |d_j(p,q) - d_0(p,q)| : (p,q) \in S_\varepsilon \} = \delta_{\varepsilon,j} \to 0,
$$

(185)

$$
Vol_{0\times 0}(S_\varepsilon) > (1 - \varepsilon) Vol_{0\times 0}(M \times M).
$$

(186)

and

$$
S_{p,\varepsilon} = \{ q \in M : (p,q) \in S_\varepsilon \},
$$

are $dvol_{g_0}$ measurable and satisfy

$$
(1 - \varepsilon) Vol_0(M) < \int_{p \in M} \frac{Vol_0(S_{p,\varepsilon})}{Vol_0(M)} dvol_{g_0}.
$$
Now we apply Egoroff’s theorem to obtain uniform convergence on a set of almost full measure.

**Proposition.** Under the hypotheses of Theorem 4.1, for every \( \varepsilon > 0 \) there exists a \( \text{dvol}_{g_0} \times \text{dvol}_{g_0} \) measurable set, \( S_\varepsilon \subset M \times M \), such that
\[
\sup \{|d_j(p,q) - d_0(p,q)| : (p,q) \in S_\varepsilon \} = \delta_{\varepsilon,j} \to 0,
\]
and
\[
\text{Vol}_{0 \times 0}(S_\varepsilon) > (1 - \varepsilon) \text{Vol}_{0 \times 0}(M \times M).
\]

\[
S_{p,\varepsilon} = \{q \in M : (p,q) \in S_\varepsilon\},
\]
are \( \text{dvol}_{g_0} \) measurable and satisfy
\[
(1 - \varepsilon) \text{Vol}_0(M) < \int_{p \in M} \frac{\text{Vol}_0(S_{p,\varepsilon})}{\text{Vol}_0(M)} \text{dvol}_{g_0}.
\]

**Lemma 4.5.** For \( W_{K,\varepsilon} = \{p : \text{Vol}_0(S_{p,\varepsilon}) > (1 - K\varepsilon) \text{Vol}_0(M)\} \)
\[
\text{Vol}_0(W_{K,\varepsilon}) > \frac{\kappa - 1}{\kappa} \text{Vol}_0(M).
\]

and
\[
|d_j(p,q) - d_0(p,q)| < \delta_{\varepsilon,j} \quad \forall p, q \in W_{K,\varepsilon}
\]
Allen-Perales-Sormani VADB to $\mathcal{VF}$ is Proven

**Lemma 4.5.** For $W_{\kappa\varepsilon} = \{p : \text{Vol}_0(S_{p,\varepsilon}) > (1-\kappa\varepsilon)\text{Vol}_0(M)\}$

\[
\text{Vol}_0(W_{\kappa\varepsilon}) > \frac{\kappa - 1}{\kappa} \text{Vol}_0(M).
\]

and $|d_j(p, q) - d_0(p, q)| < \delta_{\varepsilon,j} \quad \forall p, q \in W_{\kappa,\varepsilon}$

combined with our estimate on SWIF:

**Allen-Perales-Sormani** Let $M$ be an oriented, connected and closed manifold, $M_j = (M, g_j)$ and $M_0 = (M, g_0)$ be Riemannian manifolds with $\text{Diam}(M_j) \leq D$, $\text{Vol}_j(M_j) \leq V$ and $F_j : M_j \to M_0$ a $C^1$ diffeomorphism and distance non-increasing map:

(120) \[d_j(x, y) \geq d_0(F_j(x), F_j(y)) \quad \forall x, y \in M_j.\]

Let $W_j \subset M_j$ be a measurable set and assume that there exists a $\delta_j > 0$ so that

(121) \[d_j(x, y) \leq d_0(F_j(x), F_j(y)) + 2\delta_j \quad \forall x, y \in W_j\]

with

(122) \[\text{Vol}_j(M_j \setminus W_j) \leq V_j\]

and

(123) \[h_j \geq \sqrt{2\delta_j D + \delta_j^2}\]

then

(124) \[d_x(M_0, M_j) \leq 2V_j + h_j V.\]

completes the proof of $M_j \xrightarrow{\text{VADB}} M_\infty \implies M_j \xrightarrow{\mathcal{VF}} M_\infty$. □
SWIF limits are Integral Current Spaces

Recall Flat limits of oriented submanifolds are integral currents:

Ambrosio-Kirchheim (2000):
an integral current, $T$, on $Z$ is an integer rectifiable current s.t. $\partial T$ is also integer rectifiable.

Defn:
an integer rectifiable current, $T$, has countably many pairwise disjoint biLip charts $\phi_i: A_i \to \phi_i(A_i) \subset Z$ and weights $a_i \in \mathbb{Z}$ s.t.

$$T(f, \pi_1, \ldots, \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \phi_i) \, d(\pi_1 \circ \phi_i) \wedge \cdots \wedge d(\pi_m \circ \phi_i)$$

with mass $M(T) = ||T||(Z)$ where $||T|| = \lambda_h(T)$ and

$set(T) = \{ z \in Z | \lim \inf_{r \to 0} ||T||(B(z, r))/r^m > 0 \}$.

Key Idea:
Integral currents generalize oriented submanifolds in $Z$.

Key New Idea:
generalize oriented Riemannian Manifolds.

Sormani-Wenger Defn:
An integral current space $M = (X, d, T)$ is a metric space $(X, d)$ and integral current $T$ s.t. $set(T) = X$.

Furthermore: $M(M) = M(T)$ and $\partial M = (set(\partial T), d, \partial T)$.

Thus $X$ is countably $H_m$-rectifiable: it has countably many pairwise disjoint Lip charts $\phi_i: A_i \to X$ s.t.

$H_m(X \setminus \bigcup_{i=1}^{\infty} \phi_i(A_i)) = 0$. 

SWIF limits are Integral Current Spaces

Recall Flat limits of oriented submanifolds are integral currents: **Ambrosio-Kirchheim (2000):** an integral current, $T$, on $Z$ is an integer rectifiable current s.t. $\partial T$ is also integer rectifiable.

**Defn:** an integer rectifiable current, $T$, has cntbly many pairwise disjoint biLip charts $\varphi_i : A_i \to \varphi_i(A_i) \subset Z$ and weights $a_i \in \mathbb{Z}$ s.t.

$$T(f, \pi_1, \ldots, \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi) \, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

with mass $\mathbf{M}(T) = \|T\|(Z)$ where $\|T\| = \lambda \theta (\mathcal{H}_m \lfloor \text{set } T)$ and

$$\text{set}(T) = \{z \in Z \mid \liminf_{r \to 0} \|T\|(B(z, r))/r^m > 0\}.$$
SWIF limits are Integral Current Spaces

Recall Flat limits of oriented submanifolds are integral currents:

**Ambrosio-Kirchheim (2000):** an integral current, $T$, on $Z$ is an integer rectifiable current s.t. $\partial T$ is also integer rectifiable.

**Defn:** an integer rectifiable current, $T$, has cntbly many pairwise disjoint biLip charts $\varphi_i : A_i \to \varphi_i(A_i) \subset Z$ and weights $a_i \in \mathbb{Z}$ s.t.

$$T(f, \pi_1, \ldots, \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi) \, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

with mass $M(T) = \|T\|(Z)$ where $\|T\| = \lambda \theta (\mathcal{H}_m \sqcap \text{set } T)$ and

$$\text{set}(T) = \{ z \in Z \mid \liminf_{r \to 0} \|T\|(B(z, r))/r^m > 0 \}.$$

**Key Idea:** Integral currents generalize oriented submanifolds in $Z$. 
SWIF limits are Integral Current Spaces

Recall Flat limits of oriented submanifolds are integral currents:

**Ambrosio-Kirchheim (2000):** an integral current, $T$, on $Z$ is an integer rectifiable current s.t. $\partial T$ is also integer rectifiable.

**Defn:** an integer rectifiable current, $T$, has cntbly many pairwise disjoint biLip charts $\varphi_i : A_i \rightarrow \varphi_i(A_i) \subset Z$ and weights $a_i \in \mathbb{Z}$ s.t.

$$T(f, \pi_1, \ldots, \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

with mass $M(T) = \|T\|(Z)$ where $\|T\| = \lambda \theta (H_m \sqsubseteq \text{set } T)$ and

$$\text{set}(T) = \{z \in Z \mid \liminf_{r \to 0} \frac{\|T\|(B(z, r))}{r^m} > 0\}.$$

**Key Idea:** Integral currents generalize oriented submanifolds in $Z$.

**Key New Idea:** generalize oriented Riemannian Manifolds.
SWIF limits are Integral Current Spaces

Recall Flat limits of oriented submanifolds are integral currents: Ambrosio-Kirchheim (2000): an integral current, $T$, on $Z$ is an integer rectifiable current s.t. $\partial T$ is also integer rectifiable.

Defn: an integer rectifiable current, $T$, has cntbly many pairwise disjoint biLip charts $\varphi_i : A_i \to \varphi_i(A_i) \subset Z$ and weights $a_i \in \mathbb{Z}$ s.t.

$$T(f, \pi_1, ..., \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

with mass $M(T) = \|T\|(Z)$ where $\|T\| = \lambda \theta(H_m \downarrow \text{set } T)$ and

$$\text{set}(T) = \{ z \in Z | \liminf_{r \to 0} \|T\|(B(z, r))/r^m > 0 \}.$$ 

Key Idea: Integral currents generalize oriented submanifolds in $Z$.

Key New Idea: generalize oriented Riemannian Manifolds.

Sormani-Wenger Defn: An integral current space $M = (X, d, T)$ is a metric space $(X, d)$ and integral current $T$ s.t. $\text{set}(T) = X$. 
SWIF limits are Integral Current Spaces

Recall Flat limits of oriented submanifolds are integral currents: **Ambrosio-Kirchheim (2000):** an integral current, $T$, on $Z$ is an integer rectifiable current s.t. $\partial T$ is also integer rectifiable.

**Defn:** an integer rectifiable current, $T$, has cntbly many pairwise disjoint biLip charts $\varphi_i : A_i \to \varphi_i(A_i) \subset Z$ and weights $a_i \in \mathbb{Z}$ s.t.

$$T(f, \pi_1, ..., \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

with mass $M(T) = \| T \|(Z)$ where $\| T \| = \lambda \theta (\mathcal{H}_m \sqsubset \text{set } T)$ and

$$\text{set}(T) = \{ z \in Z \mid \liminf_{r \to 0} \| T \|(B(z, r))/r^m > 0 \}.$$

**Key Idea:** Integral currents generalize oriented submanifolds in $Z$.

**Key New Idea:** generalize oriented Riemannian Manifolds.

**Sormani-Wenger Defn:** An integral current space $M = (X, d, T)$ is a metric space $(X, d)$ and integral current $T$ s.t. set$(T) = X$.

Furthermore: $M(M) = M(T)$ and $\partial M = (\text{set}(\partial T), d, \partial T)$. 
SWIF limits are Integral Current Spaces

Recall Flat limits of oriented submanifolds are integral currents: 
**Ambrosio-Kirchheim (2000):** an integral current, $T$, on $Z$ is an integer rectifiable current s.t. $\partial T$ is also integer rectifiable. 

**Defn:** an integer rectifiable current, $T$, has cntbly many pairwise disjoint biLip charts $\varphi_i : A_i \to \varphi_i(A_i) \subset Z$ and weights $a_i \in \mathbb{Z}$ s.t.

$$T(f, \pi_1, ..., \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

with mass $M(T) = \|T\|(Z)$ where $\|T\| = \lambda \theta (H_m \perp \text{set} T)$ and

$$\text{set}(T) = \{ z \in Z | \liminf_{r \to 0} \|T\|(B(z, r))/r^m > 0 \}.$$ 

**Key Idea:** Integral currents generalize oriented submanifolds in $Z$. 

**Key New Idea:** generalize oriented Riemannian Manifolds. 

**Sormani-Wenger Defn:** An integral current space $M = (X, d, T)$ is a metric space $(X, d)$ and integral current $T$ s.t. set$(T) = X$. 

Furthermore: $M(M) = M(T)$ and $\partial M = (\text{set}(\partial T), d, \partial T)$. 

Thus $X$ is cntbly $H^m$ rectifiable: it has cntbly many pairwise disjoint Lip charts $\varphi_i : A_i \to X$ s.t. $H^m(X \setminus \bigcup_{i=1}^{\infty} \varphi_i(A_i)) = 0$. 
Oriented Riemannian Manifolds and Integral Current Spaces

An oriented Riemannian manifold \((M^m, g)\) is a metric space \((M, d_M)\) with a smooth collection of charts 

\[ d_M(p, q) = \inf \{ L_g(C) : C: [0, 1] \to M, C(0) = p, C(1) = q \} \] 

where 

\[ L_g(C) = \int_0^1 g(C'(s), C'(s))^{1/2} ds \]

\[ \text{Vol}(U) = H_m(U) \]

is the Hausdorff measure.

An integral current space \(M = (X, d, T)\) is a metric space \((X, d)\) and integral current \(T\) s.t. \(\text{set}(T) = X\) where 

\[ \text{set}(T) = \{ z \in Z | \lim \inf r \to 0 ||T||(B(z, r))/r^m > 0 \} \]

So it has a countable collection of biLipschitz charts that are oriented and weighted by \(\theta: M \to \mathbb{Z}\).

The mass \(M(U) = ||T||(U)\) has \(||T|| = \theta \lambda H_m\).

It might not be connected and might not have any geodesics.

Its boundary is \(\partial M = (\text{set}(\partial T), d, \partial T)\).

A compact oriented manifold \((M^m, g)\) is an integral current space \((M, d_M, [\cdot])\) with weight \(\theta = 1\) and 

\[ M(U) = \text{Vol}(U) = H_m(U) \] 

Its boundary \((\partial M, d_M, [\cdot])\) has the restricted distance \(d_M\).
Oriented Riemannian Manifolds and Integral Current Spaces

An oriented Riemannian manifold \((M^m, g)\) is a metric space \((M, d_M)\) with a smooth collection of charts

\[ d_M(p, q) = \inf \{ L_g(C) : C : [0, 1] \to M, \; C(0) = p, \; C(1) = q \} \]

where \( L_g(C) = \int_0^1 g(C'(s), C'(s))^{1/2} \, ds \)

\( \text{Vol}(U) = \mathcal{H}^m(U) \) is the Hausdorff measure.
Oriented Riemannian Manifolds and Integral Current Spaces

An oriented Riemannian manifold \((M^m, g)\) is a metric space \((M, d_M)\) with a smooth collection of charts

\[
d_M(p, q) = \inf \{ L_g(C) : \ C : [0, 1] \to M, \ C(0) = p, \ C(1) = q \}
\]

where \(L_g(C) = \int_0^1 g(C'(s), C'(s))^{1/2} \, ds\)

\[
Vol(U) = \mathcal{H}^m(U) \text{ is the Hausdorff measure.}
\]

An integral current space \(M = (X, d, T)\) is a metric space \((X, d)\) and integral current \(T\) s.t. set\((T) = X\) where

\[
\text{set}(T) = \{ z \in Z | \lim \inf_{r \to 0} \|T\|(B(z, r))/r^m > 0 \}
\]

So it has a countable collection of biLipschitz charts
Oriented Riemannian Manifolds and Integral Current Spaces

An oriented Riemannian manifold \((M^m, g)\) is a metric space \((M, d_M)\) with a smooth collection of charts
\[
d_M(p, q) = \inf \{ L_g(C) : C : [0, 1] \to M, \ C(0) = p, \ C(1) = q \}
\]
where \(L_g(C) = \int_0^1 g(C'(s), C'(s))^{1/2} \, ds\)
\[
\text{Vol}(U) = \mathcal{H}^m(U) \text{ is the Hausdorff measure.}
\]
An integral current space \(M = (X, d, T)\) is a metric space \((X, d)\) and integral current \(T\) s.t. \(\text{set}(T) = X\) where
\[
\text{set}(T) = \{ z \in Z \mid \liminf_{r \to 0} \| T \|(B(z, r))/r^m > 0 \}
\]
So it has a countable collection of biLipschitz charts that are oriented and weighted by \(\theta : M \to \mathbb{Z}\).
Oriented Riemannian Manifolds and Integral Current Spaces

An oriented Riemannian manifold \((M^m, g)\) is a metric space \((M, d_M)\) with a smooth collection of charts
\[
d_M(p, q) = \inf \{ L_g(C) : C : [0, 1] \to M, \ C(0) = p, \ C(1) = q \}
\]
where
\[
L_g(C) = \int_0^1 g(C'(s), C'(s))^{1/2} \, ds
\]

\[
Vol(U) = \mathcal{H}^m(U)
\]
is the Hausdorff measure.

An integral current space \(M = (X, d, T)\) is a metric space \((X, d)\) and integral current \(T\) s.t. \(\text{set}(T) = X\) where
\[
\text{set}(T) = \{ z \in Z | \liminf_{r \to 0} ||T||(B(z, r))/r^m > 0 \}
\]
So it has a countable collection of biLipschitz charts
that are oriented and weighted by \(\theta : M \to \mathbb{Z}\)
The mass \(M(U) = ||T||(U)\) has \(||T|| = \theta \lambda \mathcal{H}^m\).
Oriented Riemannian Manifolds and Integral Current Spaces

An oriented Riemannian manifold \((M^m, g)\) is a metric space \((M, d_M)\) with a smooth collection of charts

\[
d_M(p, q) = \inf \{ L_g(C) : C : [0, 1] \to M, \; C(0) = p, \; C(1) = q \}
\]

where \(L_g(C) = \int_0^1 g(C'(s), C'(s))^{{1/2}} \, ds\)

\[
Vol(U) = \mathcal{H}^m(U)
\]

is the Hausdorff measure.

An integral current space \(M = (X, d, T)\) is a metric space \((X, d)\) and integral current \(T\) s.t. \(\text{set}(T) = X\) where

\[
\text{set}(T) = \{ z \in Z | \liminf_{r \to 0} \frac{\| T \|(B(z, r))}{r^m} > 0 \}
\]

So it has a countable collection of biLipschitz charts that are oriented and weighted by \(\theta : M \to \mathbb{Z}\)

The mass \(M(U) = \| T \|(U)\) has \(\| T \| = \theta \lambda \mathcal{H}^m\).

It might not be connected and might not have any geodesics.
Oriented Riemannian Manifolds and Integral Current Spaces

An oriented Riemannian manifold \((M^m, g)\) is a metric space \((M, d_M)\) with a smooth collection of charts
\[
d_M(p, q) = \inf\{L_g(C) : C : [0, 1] \to M, \ C(0) = p, \ C(1) = q\}
\]

where \(L_g(C) = \int_0^1 g(C'(s), C'(s))^{1/2} \, ds\)

\[Vol(U) = \mathcal{H}^m(U)\] is the Hausdorff measure.

An integral current space \(M = (X, d, T)\) is a metric space \((X, d)\) and integral current \(T\) s.t. \(\text{set}(T) = X\) where
\[
\text{set}(T) = \{z \in Z | \liminf_{r \to 0} \|T\|(B(z, r))/r^m > 0\}
\]

So it has a countable collection of biLipschitz charts

that are oriented and weighted by \(\theta : M \to \mathbb{Z}\)

The mass \(M(U) = \|T\|(U)\) has \(\|T\| = \theta \lambda \mathcal{H}^m\).

It might not be connected and might not have any geodesics.

Its boundary is \(\partial M = (\text{set}(\partial T), d, \partial T)\).
An oriented Riemannian manifold \((M^m, g)\) is a metric space \((M, d_M)\) with a smooth collection of charts
\[
d_M(p, q) = \inf \{ L_g(C) : C : [0, 1] \to M, C(0) = p, C(1) = q \}
\]
where \(L_g(C) = \int_0^1 g(C'(s), C'(s))^{1/2} \, ds\)
\[
\operatorname{Vol}(U) = \mathcal{H}^m(U)
\]
is the Hausdorff measure.

An integral current space \(M = (X, d, T)\) is a metric space \((X, d)\) and integral current \(T\) s.t. \(\operatorname{set}(T) = X\) where
\[
\operatorname{set}(T) = \{ z \in Z \mid \liminf_{r \to 0} ||T||((B(z, r))/r^m > 0 \}
\]

So it has a countable collection of biLipschitz charts that are oriented and weighted by \(\theta : M \to \mathbb{Z}\).

The mass \(\mu(U) = ||T||(U)\) has \(||T|| = \theta \lambda \mathcal{H}^m\).

It might not be connected and might not have any geodesics.

Its boundary is \(\partial M = (\operatorname{set}(\partial T), d, \partial T)\).
Oriented Riemannian Manifolds and Integral Current Spaces

An oriented Riemannian manifold \((M^m, g)\) is a metric space \((M, d_M)\) with a smooth collection of charts

\[
d_M(p, q) = \inf \{ L_g(C) : C : [0, 1] \to M, \ C(0) = p, \ C(1) = q \}
\]

where \(L_g(C) = \int_0^1 g(C'(s), C'(s))^{1/2} \, ds\)

\[
\text{Vol}(U) = \mathcal{H}^m(U)
\]

is the Hausdorff measure.

An integral current space \(M = (X, d, T)\) is a metric space \((X, d)\) and integral current \(T\) s.t. \(\text{set}(T) = X\) where

\[
\text{set}(T) = \{ z \in Z | \lim \inf_{r \to 0} ||T||(B(z, r))/r^m > 0 \}
\]

So it has a countable collection of biLipschitz charts that are oriented and weighted by \(\theta : M \to \mathbb{Z}\)

The mass \(M(U) = ||T||(U)\) has \(||T|| = \theta \chi \mathcal{H}^m\).

It might not be connected and might not have any geodesics.

Its boundary is \(\partial M = (\text{set}(\partial T), d, \partial T)\).

A compact oriented manifold \((M^m, g)\) is an integral current space \((M, d_M, [[M]])\) with weight \(\theta = 1\) and \(M(U) = \text{Vol}(U) = \mathcal{H}^m(U)\).
Oriented Riemannian Manifolds and Integral Current Spaces

An oriented Riemannian manifold \((M^m, g)\) is a metric space \((M, d_M)\) with a smooth collection of charts

\[
d_M(p, q) = \inf \{ L_g(C) : C : [0, 1] \rightarrow M, \ C(0) = p, \ C(1) = q \}
\]

where \(L_g(C) = \int_0^1 g(C'(s), C'(s))^{1/2} \, ds\)

\[
Vol(U) = \mathcal{H}^m(U)
\]

is the Hausdorff measure.

An integral current space \(M = (X, d, T)\) is a metric space \((X, d)\) and integral current \(T\) s.t. \(\text{set}(T) = X\) where

\[
\text{set}(T) = \{ z \in Z | \liminf_{r \to 0} \| T \|(B(z, r))/r^m > 0 \}
\]

So it has a countable collection of biLipschitz charts that are oriented and weighted by \(\theta : M \rightarrow \mathbb{Z}\)

The mass \(M(U) = \| T \|(U)\) has \(\| T \| = \theta \lambda \mathcal{H}^m\).

It might not be connected and might not have any geodesics.

Its boundary is \(\partial M = (\text{set}(\partial T), d, \partial T)\).

A compact oriented manifold \((M^m, g)\) is an integral current space \((M, d_M, [[M]])\) with weight \(\theta = 1\) and \(M(U) = \text{Vol}(U) = \mathcal{H}^m(U)\).

Its boundary \((\partial M, d_M, [[\partial M]])\) has the restricted distance \(d_M\).
Sormani-Wenger: Intrinsic Flat Distance

between integral current spaces \( M_i^m = (X_i, d_i, T_i) \) is:

\[
d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ \bar{d}_F(\varphi_1#T_1, \varphi_2#T_2) \mid \varphi_i : M_i^m \to Z \right\}
\]

where \( \varphi_#T(f, \pi_1, ..., \pi_m) = T(f \circ \varphi, \pi_1 \circ \varphi, ..., \pi_m \circ \varphi) \),
Sormani-Wenger: Intrinsic Flat Distance

between integral current spaces $M_i^m = (X_i, d_i, T_i)$ is:

$$d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d_Z^F (\varphi_1\# T_1, \varphi_2\# T_2) \mid \varphi_i : M_i^m \to Z \right\}$$

where $\varphi\# T(f, \pi_1, \ldots, \pi_m) = T(f \circ \varphi, \pi_1 \circ \varphi, \ldots, \pi_m \circ \varphi)$, and

where the infimum is taken over all complete metric spaces, $Z$,

and over all distance preserving maps $\varphi_i : M_i^m \to Z$.

Here: $d_Z^F (\varphi_1\# T_1, \varphi_2\# T_2)$ is the Federer-Fleming Flat dist

$$= \inf \left\{ \begin{array}{c} \text{M (A)} + \text{M (B)} : \text{A} + \partial \text{B} = \varphi_1\# T_1 - \varphi_2\# T_2 \end{array} \right\}$$

$$= \inf \left\{ \begin{array}{c} \text{M (A)} + \text{M (B)} : \text{A} + \partial \text{B} = \varphi_1\# T_1 - \varphi_2\# T_2 \end{array} \right\}$$
Implications of SWIF Convergence [SW-JDG]

**Defn:** For any pair of integral current spaces,

\[ d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d_F^Z(\varphi_1#T_1, \varphi_2#T_2) \mid \varphi_i : M_i^m \to Z \right\} \]

where the inf over complete \( Z \) and dist. pres. \( \varphi_i : M_i^m \to Z \).

**Thm:** The infimum is achieved, so we can choose

\[ Z' = \text{set}(A) \cup \text{set}(B) \]

which is separable and rectifiable.
Implications of SWIF Convergence [SW-JDG]

**Defn:** For any pair of integral current spaces,

\[ d_{SWIF}(M^m_1, M^m_2) = \inf \left\{ d^Z_F(\varphi_1\#T_1, \varphi_2\#T_2) \mid \varphi_i : M^m_i \to Z \right\} \]

where the inf over complete \( Z \) and dist. pres. \( \varphi_i : M^m_i \to Z \).

**Thm:** The infimum is achieved, so we can choose \( Z' = \text{set}(A) \cup \text{set}(B) \) which is separable and rectifiable.

**Thm:** If \( M_j = (X_j, d_j, T_j) \xrightarrow{SWIF} M_0 = (X_0, d_0, T_0) \)
Implications of SWIF Convergence [SW-JDG]

**Defn:** For any pair of integral current spaces,

\[ d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d_Z^F(\varphi_1 # T_1, \varphi_2 # T_2) \mid \varphi_i : M_i^m \to Z \right\} \]

where the inf over complete \( Z \) and dist. pres. \( \varphi_i : M_i^m \to Z \).

**Thm:** The infimum is achieved, so we can choose

\[ Z' = \text{set}(A) \cup \text{set}(B) \]

which is separable and rectifiable.

**Thm:** If \( M_j = (X_j, d_j, T_j) \xrightarrow{SWIF} M_0 = (X_0, d_0, T_0) \)

\[ \exists \text{ complete separable } Z \text{ and dist. pres. } \varphi_j : X_j \to Z \]

s.t. \( d_Z^F(\varphi_j # T_j, \varphi_0 # T_0) \to 0 \) and \( \varphi_j # T_j(\omega) \to \varphi_0 # T_0(\omega) \).
Implications of SWIF Convergence [SW-JDG]

Defn: For any pair of integral current spaces,

\[ d_{SWIF}(M^m_1, M^m_2) = \inf \left\{ d^Z_F(\varphi_1\# T_1, \varphi_2\# T_2) \mid \varphi_i : M^m_i \to Z \right\} \]

where the inf over complete \( Z \) and dist. pres. \( \varphi_i : M^m_i \to Z \).

Thm: The infimum is achieved, so we can choose

\[ Z' = \text{set}(A) \cup \text{set}(B) \]

which is separable and rectifiable.

Thm: If \( M_j = (X_j, d_j, T_j) \overset{\text{SWIF}}{\longrightarrow} M_0 = (X_0, d_0, T_0) \)

\( \exists \) complete separable \( Z \) and dist. pres. \( \varphi_j : X_j \to Z \)

s.t. \( d^Z_F(\varphi_j\# T_j, \varphi_0\# T_0) \to 0 \) and \( \varphi_j\# T_j(\omega) \to \varphi_0\# T_0(\omega) \).

Thus by Ambrosio-Kirchheim Theory:
Implications of SWIF Convergence [SW-JDG]

**Defn:** For any pair of integral current spaces,

\[ d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d_Z^F(\varphi_1#T_1, \varphi_2#T_2) \mid \varphi_i : M_i^m \to Z \right\} \]

where the inf over complete $Z$ and dist. pres. $\varphi_i : M_i^m \to Z$.

**Thm:** The infimum is achieved, so we can choose $Z' = \text{set}(A) \cup \text{set}(B)$ which is separable and rectifiable.

**Thm:** If $M_j = (X_j, d_j, T_j) \xrightarrow{SWIF} M_0 = (X_0, d_0, T_0)$

\[ \exists \text{ complete separable } Z \text{ and dist. pres. } \varphi_j : X_j \to Z \]

s.t. $d_Z^F(\varphi_j#T_j, \varphi_0#T_0) \to 0$ and $\varphi_j#T_j(\omega) \to \varphi_0#T_0(\omega)$.

**Thus by Ambrosio-Kirchheim Theory:**

\[ M_j \xrightarrow{SWIF} M_\infty \implies \partial M_j \xrightarrow{SWIF} \partial M_\infty \]
Implications of SWIF Convergence [SW-JDG]

**Defn:** For any pair of integral current spaces,

\[ d_{SWIF}(M^m_1, M^m_2) = \inf \left\{ \frac{d}{dF}(\varphi_1\#T_1, \varphi_2\#T_2) \mid \varphi_i : M^m_i \to Z \right\} \]

where the inf over complete \( Z \) and dist. pres. \( \varphi_i : M^m_i \to Z \).

**Thm:** The infimum is achieved, so we can choose
\[ Z' = \text{set}(A) \cup \text{set}(B) \]
which is separable and rectifiable.

**Thm:** If \( M_j = (X_j, d_j, T_j) \xrightarrow{SWIF} M_0 = (X_0, d_0, T_0) \)
\[ \exists \text{ complete separable } Z \text{ and dist. pres. } \varphi_j : X_j \to Z \]
s.t. \( \frac{d}{dF}(\varphi_j\#T_j, \varphi_0\#T_0) \to 0 \) and \( \varphi_j\#T_j(\omega) \to \varphi_0\#T_0(\omega) \).

**Thus by Ambrosio-Kirchheim Theory:**

\[ M_j \xrightarrow{SWIF} M_\infty \implies \partial M_j \xrightarrow{SWIF} \partial M_\infty \]

\[ M_j \xrightarrow{SWIF} M_\infty \implies \lim\inf_{j \to \infty} M(M_j) \geq M(M_\infty) \]

**Thm [Sor-ArzAsc]:** For any \( p \in M_\infty \) there exists \( p_j \in M_j \) s.t.
\[ d_Z(\varphi_j(p_j), \varphi_\infty(p)) \to 0. \]
Theorem [Sor-ArzAsc] Suppose $M_i = (X_i, d_i, T_i)$ are integral current spaces for $i \in \{1, 2, ..., \infty\}$ and $M_i \xrightarrow{\mathcal{F}} M_\infty$ and $F_i : X_i \to W$ are Lipschitz maps into a compact metric space $W$ with

\begin{equation}
\text{Lip}(F_i) \leq K,
\end{equation}

then a subsequence converges to a Lipschitz map $F_\infty : X_\infty \to W$ with

\begin{equation}
\text{Lip}(F_\infty) \leq K.
\end{equation}

More specifically, there exists isometric embeddings of the subsequence, $\varphi_i : X_i \to Z$, such that $d_F^Z(\varphi_i\#T_i, \varphi_\infty\#T_\infty) \to 0$ and for any sequence $p_i \in X_i$ converging to $p \in X_\infty$,

\begin{equation}
\text{d}_Z(\varphi_i(p_i), \varphi_\infty(p)) \to 0,
\end{equation}

one has converging images,

\begin{equation}
\text{d}_W(F_i(p_i), F_\infty(p)) \to 0.
\end{equation}
Balls and SWIF Limits [Sormani-ArzAsc]:

**Thm:** If $B(p, r)$ is a ball in an integral current space $M$ then $(B(p, r), d_M, [[B(p, r)]]))$ is an integral current space for a.e. $r > 0$. 

**Thm [SW]:** If $M_j \xrightarrow{SWIF} M_0 = (x_0, d_0, T_0)$ then there exist complete separable $Z$ and dist. pres. $\phi_j : X_j \to Z$ such that $d_Z(\phi_j(T_j), \phi_0(T_0)) \to 0$ and $\phi_j(T_j)(\omega) \to \phi_0(T_0)(\omega)$.

**Thm:** If $p_0 \in M_0$ and $M_j \xrightarrow{SWIF} M_0$, then there exist $p_j \in M_j$ such that $d_Z(\phi_j(p_j), \phi_0(p_0)) \to 0$.

Furthermore:
- For a.e. $r > 0$ there exists a subsequence $j_k$ such that $(B(p_{j_k}, r), d_M, [[B(p_{j_k}, r)]]) \xrightarrow{SWIF} (B(p_0, r), d_M, [[B(p_0, r)])$.

**Coro:** $\liminf_{j \to \infty} M(B(p_j, r)) \geq M(B(p_0, r))$.

**Coro:** $\partial B(p_j, r) \to \partial B(p_0, r)$.

**Coro:** $\text{FillVol}(\partial B(p_j, r)) \to \text{FillVol}(\partial B(p_0, r))$.

**Coro:** $\text{Diam}(M_0) \leq \liminf_{j \to \infty} \text{Diam}(M_j)$.

**Thm:** If $M_m j \xrightarrow{SWIF} M \neq 0$ then there exists $N_j \subset M_j$ such that $N_j \xrightarrow{GH} M$ SWIF and $\liminf_{j \to \infty} M(N_j) \geq M(M_{SWIF})$.

The proofs of the above use the Ambrosio-Kirchheim Slicing Thm.
Balls and SWIF Limits [Sormani-ArzAsc]:

**Thm:** If $B(p, r)$ is a ball in an integral current space $M$ then $(B(p, r), d_M, [[B(p, r)]]))$ is an integral current space for a.e. $r > 0$.

**Thm [SW]:** If $M_j = (X_j, d_j, T_j) \xrightarrow{\text{SWIF}} M_0 = (x_0, d_0, T_0)$

$\exists$ complete separable $Z$ and dist. pres. $\varphi_j : X_j \rightarrow Z$ such that

$d^Z_F(\varphi_j\#T_j, \varphi_0\#T_0) \rightarrow 0$ and $\varphi_j\#T_j(\omega) \rightarrow \varphi_0\#T_0(\omega)$.
Balls and SWIF Limits [Sormani-ArzAsc]:

**Thm:** If $B(p, r)$ is a ball in an integral current space $M$ then $(B(p, r), d_M, [[B(p, r)]])$ is an integral current space for a.e. $r > 0$.

**Thm [SW]:** If $M_j = (X_j, d_j, T_j) \xrightarrow{SWIF} M_0 = (x_0, d_0, T_0)$

$\exists$ complete separable $Z$ and dist. pres. $\varphi_j : X_j \to Z$ such that

$d^Z_F(\varphi_j\# T_j, \varphi_0\# T_0) \to 0$ and $\varphi_j\# T_j(\omega) \to \varphi_0\# T_0(\omega)$.

**Thm:** If $p_0 \in M_0$ and $M_j \xrightarrow{SWIF} M_0$, then
Balls and SWIF Limits [Sormani-ArzAsc]:

**Thm:** If $B(p, r)$ is a ball in an integral current space $M$ then
$(B(p, r), d_M, [[B(p, r)]])$ is an integral current space for a.e. $r > 0$.

**Thm [SW]:** If $M_j = (X_j, d_j, T_j) \xrightarrow{\text{SWIF}} M_0 = (x_0, d_0, T_0)$

exists complete separable $Z$ and dist. pres. $\varphi_j : X_j \to Z$ such that

$d^Z_F(\varphi_j \# T_j, \varphi_0 \# T_0) \to 0$ and $\varphi_j \# T_j(\omega) \to \varphi_0 \# T_0(\omega)$.

**Thm:** If $p_0 \in M_0$ and $M_j \xrightarrow{\text{SWIF}} M_0$, then

$\exists p_j \in M_j$ such that $d_Z(\varphi_j(p_j), \varphi_0(p_0)) \to 0$. 

The proofs of the above use the Ambrosio-Kirchheim Slicing Thm.
Balls and SWIF Limits [Sormani-ArzAsc]:

Thm: If $B(p, r)$ is a ball in an integral current space $M$ then $(B(p, r), d_M, [[B(p, r)]])$ is an integral current space for a.e. $r > 0$.

Thm [SW]: If $M_j = (X_j, d_j, T_j) \xrightarrow{\text{SWIF}} M_0 = (x_0, d_0, T_0)$

exists complete separable $Z$ and dist. pres. $\varphi_j : X_j \to Z$ such that

$d_Z^F(\varphi_j# T_j, \varphi_0# T_0) \to 0$ and $\varphi_j# T_j(\omega) \to \varphi_0# T_0(\omega)$.

Thm: If $p_0 \in M_0$ and $M_j \xrightarrow{\text{SWIF}} M_0$, then

exists $p_j \in M_j$ such that $d_Z(\varphi_j(p_j), \varphi_0(p_0)) \to 0$.

Furthermore: For a.e. $r > 0$ exists subsequence $j_k$ such that

$(B(p_{j_k}, r), d_M, [[B(p_{j_k}, r)]]) \xrightarrow{\text{SWIF}} (B(p_0, r), d_M, [[B(p_0, r)]])$. 
Balls and SWIF Limits [Sormani-ArzAsc]:

**Thm:** If $B(p, r)$ is a ball in an integral current space $M$ then $(B(p, r), d_M, [[B(p, r)]]))$ is an integral current space for a.e. $r > 0$.

**Thm [SW]:** If $M_j = (X_j, d_j, T_j) \xrightarrow{\text{SWIF}} M_0 = (x_0, d_0, T_0)$

$\exists$ complete separable $Z$ and dist. pres. $\varphi_j : X_j \rightarrow Z$ such that

$$d_Z(\varphi_j#T_j, \varphi_0#T_0) \rightarrow 0 \text{ and } \varphi_j#T_j(\omega) \rightarrow \varphi_0#T_0(\omega).$$

**Thm:** If $p_0 \in M_0$ and $M_j \xrightarrow{\text{SWIF}} M_0$, then

$\exists p_j \in M_j$ such that $d_Z(\varphi_j(p_j), \varphi_0(p_0)) \rightarrow 0$.

**Furthermore:** For a.e. $r > 0 \exists$ subsequence $j_k$ such that

$(B(p_{jk}, r), d_M, [[B(p_{jk}, r)]])) \xrightarrow{\text{SWIF}} (B(p_0, r), d_M, [[B(p_0, r)]]))$.

**Coro:** $\lim \inf_{j \rightarrow \infty} M(B(p_j, r)) \geq M(B(p_0, r))$. 
Balls and SWIF Limits [Sormani-ArzAsc]:

**Thm:** If $B(p, r)$ is a ball in an integral current space $M$ then $(B(p, r), d_M, [[B(p, r)]]))$ is an integral current space for a.e. $r > 0$.

**Thm [SW]:** If $M_j = (X_j, d_j, T_j) \xrightarrow{\text{SWIF}} M_0 = (x_0, d_0, T_0)$

$\exists$ complete separable $Z$ and dist. pres. $\varphi_j : X_j \rightarrow Z$ such that

$d_Z^F(\varphi_j#T_j, \varphi_0#T_0) \rightarrow 0$ and $\varphi_j#T_j(\omega) \rightarrow \varphi_0#T_0(\omega)$.

**Thm:** If $p_0 \in M_0$ and $M_j \xrightarrow{\text{SWIF}} M_0$, then

$\exists p_j \in M_j$ such that $d_Z(\varphi_j(p_j), \varphi_0(p_0)) \rightarrow 0$.

**Furthermore:** For a.e. $r > 0$ $\exists$ subsequence $j_k$ such that

$(B(p_{j_k}, r), d_M, [[B(p_{j_k}, r)]])) \xrightarrow{\text{SWIF}} (B(p_0, r), d_M, [[B(p_0, r)]]))$.

**Coro:** $\liminf_{j \rightarrow \infty} \mathcal{M}(B(p_j, r)) \geq \mathcal{M}(B(p_0, r))$.

**Coro:** $\partial B(p_j, r) \rightarrow \partial B(p_0, r)$. 

The proofs of the above use the Ambrosio-Kirchheim Slicing Thm.
Balls and SWIF Limits [Sormani-ArzAsc]:

**Thm:** If \( B(p, r) \) is a ball in an integral current space \( M \) then \( (B(p, r), d_M, [[B(p, r)]]) \) is an integral current space for a.e. \( r > 0 \).

**Thm [SW]:** If \( M_j = (X_j, d_j, T_j) \xrightarrow{SWIF} M_0 = (x_0, d_0, T_0) \)
\( \exists \) complete separable \( Z \) and dist. pres. \( \varphi_j : X_j \to Z \) such that
\[
d_Z^F(\varphi_j\#T_j, \varphi_0\#T_0) \to 0 \quad \text{and} \quad \varphi_j\#T_j(\omega) \to \varphi_0\#T_0(\omega).
\]

**Thm:** If \( p_0 \in M_0 \) and \( M_j \xrightarrow{SWIF} M_0 \), then
\( \exists p_j \in M_j \) such that \( d_Z(\varphi_j(p_j), \varphi_0(p_0)) \to 0 \).

**Furthermore:** For a.e. \( r > 0 \) \( \exists \) subsequence \( j_k \) such that
\( (B(p_{j_k}, r), d_M, [[B(p_{j_k}, r)]]) \xrightarrow{SWIF} (B(p_0, r), d_M, [[B(p_0, r)])] \).

**Coro:** \( \liminf_{j \to \infty} M(B(p_j, r)) \geq M(B(p_0, r)) \).

**Coro:** \( \partial B(p_j, r) \to \partial B(p_0, r) \).

**Coro:** \( \text{FillVol}(\partial B(p_j, r)) \to \text{FillVol}(\partial B(p_0, r)). \)
**Balls and SWIF Limits [Sormani-ArzAsc]:**

**Thm:** If $B(p, r)$ is a ball in an integral current space $M$ then $(B(p, r), d_M, [[B(p, r)]]))$ is an integral current space for a.e. $r > 0$.

**Thm [SW]:** If $M_j = (X_j, d_j, T_j) \xrightarrow{SWIF} M_0 = (x_0, d_0, T_0)$  
$\exists$ complete separable $Z$ and dist. pres. $\varphi_j : X_j \rightarrow Z$ such that  
$d^Z_F(\varphi_j\#T_j, \varphi_0\#T_0) \rightarrow 0$ and $\varphi_j\#T_j(\omega) \rightarrow \varphi_0\#T_0(\omega)$.

**Thm:** If $p_0 \in M_0$ and $M_j \xrightarrow{SWIF} M_0$, then  
$\exists p_j \in M_j$ such that $d_Z(\varphi_j(p_j), \varphi_0(p_0)) \rightarrow 0$.

**Furthermore:** For a.e. $r > 0$ $\exists$ subsequence $j_k$ such that  
$(B(p_{j_k}, r), d_M, [[B(p_{j_k}, r)]])) \xrightarrow{SWIF} (B(p_0, r), d_M, [[B(p_0, r)]]))$.

**Coro:** $\liminf_{j \rightarrow \infty} M(B(p_j, r)) \geq M(B(p_0, r))$.

**Coro:** $\partial B(p_j, r) \rightarrow \partial B(p_0, r)$.

**Coro:** $\text{FillVol}(\partial B(p_j, r)) \rightarrow \text{FillVol}(\partial B(p_0, r))$.

**Coro:** $\text{Diam}(M_0) \leq \liminf_{j \rightarrow \infty} \text{Diam}(M_j)$.

The proofs of the above use the Ambrosio-Kirchheim Slicing Thm.
Balls and SWIF Limits [Sormani-ArzAsc]:

**Thm:** If \( B(p, r) \) is a ball in an integral current space \( M \) then \((B(p, r), d_M, [[B(p, r)]])) is an integral current space for a.e. \( r > 0 \).

**Thm [SW]:** If \( M_j = (X_j, d_j, T_j) \xrightarrow{SWIF} M_0 = (x_0, d_0, T_0) \) there exist complete separable \( Z \) and dist. pres. \( \varphi_j : X_j \to Z \) such that \( d_Z^F(\varphi_j\#T_j, \varphi_0\#T_0) \to 0 \) and \( \varphi_j\#T_j(\omega) \to \varphi_0\#T_0(\omega) \).

**Thm:** If \( p_0 \in M_0 \) and \( M_j \xrightarrow{SWIF} M_0 \), then there exist \( p_j \in M_j \) such that \( d_Z(\varphi_j(p_j), \varphi_0(p_0)) \to 0 \).

**Furthermore:** For a.e. \( r > 0 \) there exist a subsequence \( j_k \) such that \((B(p_{j_k}, r), d_M, [[B(p_{j_k}, r)]])) \xrightarrow{SWIF} (B(p_0, r), d_M, [[B(p_0, r)]]))\).

**Coro:** \( \liminf_{j \to \infty} M(B(p_j, r)) \geq M(B(p_0, r)) \).

**Coro:** \( \partial B(p_j, r) \to \partial B(p_0, r) \).

**Coro:** \( \text{FillVol}(\partial B(p_j, r)) \to \text{FillVol}(\partial B(p_0, r)) \).

**Coro:** \( \text{Diam}(M_0) \leq \liminf_{j \to \infty} \text{Diam}(M_j) \).

**Thm:** If \( M^m_j \xrightarrow{SWIF} M_{SWIF} \neq 0^m \) then \( \exists N_j \subset M_j \) such that \( N_j \xrightarrow{GH} M_{SWIF} \) and \( \liminf_{j \to \infty} M(N_j) \geq M(M_{SWIF}) \).

The proofs of the above use the Ambrosio-Kirchheim Slicing Thm.
Balls and SWIF Limits [Sormani-ArzAsc]:

**Thm:** If \( B(p, r) \) is a ball in an integral current space \( M \) then \((B(p, r), d_M, [[B(p, r)]])\) is an integral current space for a.e. \( r > 0 \).

**Thm [SW]:** If \( M_j = (X_j, d_j, T_j) \xrightarrow{SWIF} M_0 = (x_0, d_0, T_0) \) there exists a complete separable \( Z \) and distance preserving \( \varphi_j : X_j \to Z \) such that \( d^Z_F(\varphi_j\#T_j, \varphi_0\#T_0) \to 0 \) and \( \varphi_j\#T_j(\omega) \to \varphi_0\#T_0(\omega) \).

**Thm:** If \( p_0 \in M_0 \) and \( M_j \xrightarrow{SWIF} M_0 \), then there exists \( p_j \in M_j \) such that \( d_Z(\varphi_j(p_j), \varphi_0(p_0)) \to 0 \).

**Furthermore:** For a.e. \( r > 0 \) there exists a subsequence \( j_k \) such that \((B(p_{j_k}, r), d_M, [[B(p_{j_k}, r)]]) \xrightarrow{SWIF} (B(p_0, r), d_M, [[B(p_0, r)]])\).

**Coro:** \( \liminf_{j \to \infty} M(B(p_j, r)) \geq M(B(p_0, r)) \).

**Coro:** \( \partial B(p_j, r) \to \partial B(p_0, r) \).

**Coro:** \( \text{FillVol}(\partial B(p_j, r)) \to \text{FillVol}(\partial B(p_0, r)) \).

**Coro:** \( \text{Diam}(M_0) \leq \liminf_{j \to \infty} \text{Diam}(M_j) \).

**Thm:** If \( M_j^m \xrightarrow{SWIF} M_{SWIF} \neq 0^m \) then there exists \( N_j \subset M_j \) such that \( N_j \xrightarrow{GH} M_{SWIF} \) and \( \liminf_{j \to \infty} M(N_j) \geq M(M_{SWIF}) \).

The proofs of the above use the Ambrosio-Kirchheim Slicing Thm.
Ambrosio-Kirchheim Slicing Theorem:
Given Lipschitz, $f : Z \to \mathbb{R}$, and integral current, $T$, for a.e. $s \in \mathbb{R}$
one can define the slice of $T$ by $f$ at $s$ which is an integral current
\[
<T, f, s> := -\partial (T \upharpoonright f^{-1}(s, \infty)) + (\partial T) \upharpoonright f^{-1}(s, \infty),
\]
where $S$ restricted to $U$ is $(S \upharpoonright U)(h, \pi_1, \ldots) = S(\chi_U \cdot h, \pi_1, \ldots)$. 

Ambrosio-Kirchheim Slicing Theorem:

Given Lipschitz, \( f : Z \to \mathbb{R} \), and integral current, \( T \), for a.e. \( s \in \mathbb{R} \) one can define the slice of \( T \) by \( f \) at \( s \) which is an integral current

\[
<T, f, s> := -\partial (T \upharpoonright f^{-1}(s, \infty)) + (\partial T) \upharpoonright f^{-1}(s, \infty),
\]

where \( S \) restricted to \( U \) is \((S \upharpoonright U)(h, \pi_1, ...) = S(\chi_U \cdot h, \pi_1, ...).\)

To prove it is an integral current, they prove its mass and the mass of \( \partial <T, f, s> = \langle -\partial T, f, s \rangle \) is finite for a.e. \( s \in \mathbb{R} \).
Ambrosio-Kirchheim Slicing Theorem:

Given Lipschitz, \( f : Z \to \mathbb{R} \), and integral current, \( T \), for a.e. \( s \in \mathbb{R} \), one can define the slice of \( T \) by \( f \) at \( s \) which is an integral current

\[
< T, f, s > := -\partial \left( T \llcorner f^{-1}(s, \infty) \right) + (\partial T) \llcorner f^{-1}(s, \infty),
\]

where \( S \) restricted to \( U \) is \( (S \llcorner U)(h, \pi_1, ...) = S(\chi_U \cdot h, \pi_1, ...) \).

To prove it is an integral current, they prove its mass and the mass of \( \partial < T, f, s > = < -\partial T, f, s > \) is finite for a.e. \( s \in \mathbb{R} \). In fact:

\[
\int_{s \in \mathbb{R}} M(< T, f, s >) \, ds = M(T \llcorner df) \leq \text{Lip}(f) \, M(T)
\]

where \((T \llcorner df)(h, \pi_1, ..., \pi_{m-1}) = T(h, f, \pi_1, ..., \pi_{m-1})\).
Flat Distance between Slices in $Z$

Given integral currents $T_i$ in $Z$
then we have $T_1 - T_2 = A + \partial B$
where $d_F^Z(T_1, T_2) = \mathbf{M}(A) + \mathbf{M}(B)$. 
Flat Distance between Slices in $Z$

Given integral currents $T_i$ in $Z$ then we have $T_1 - T_2 = A + \partial B$
where $d_{\mathcal{F}}(T_1, T_2) = M(A) + M(B)$.

$< T_1, f, s > - < T_2, f, s > = < A, f, s > + < \partial B, f, s >$
Flat Distance between Slices in $Z$

Given integral currents $T_i$ in $Z$
then we have $T_1 - T_2 = A + \partial B$
where $d_F^Z(T_1, T_2) = M(A) + M(B)$.

$< T_1, f, s > - < T_2, f, s > = < A, f, s > + < \partial B, f, s >$

$< T_1, f, s > - < T_2, f, s > = < A, f, s > - \partial < B, f, s >$
Flat Distance between Slices in $Z$

Given integral currents $T_i$ in $Z$
then we have $T_1 - T_2 = A + \partial B$
where $d^Z_F(T_1, T_2) = \mathbf{M}(A) + \mathbf{M}(B)$.

$< T_1, f, s > - < T_2, f, s > = < A, f, s > + < \partial B, f, s >$

$< T_1, f, s > - < T_2, f, s > = < A, f, s > - \partial < B, f, s >$

$d^Z_F(< T_1, f, s >, < T_2, f, s >) \leq \mathbf{M}(< A, f, s >) + \mathbf{M}(< B, f, s >)$
Flat Distance between Slices in $Z$

Given integral currents $T_i$ in $Z$ then we have $T_1 - T_2 = A + \partial B$ where $d_F^Z(T_1, T_2) = \mathbf{M}(A) + \mathbf{M}(B)$.

\[
< T_1, f, s > - < T_2, f, s > = < A, f, s > + < \partial B, f, s >
\]
\[
< T_1, f, s > - < T_2, f, s > = < A, f, s > - \partial < B, f, s >
\]
\[
d_F^Z(< T_1, f, s >, < T_2, f, s >) \leq \mathbf{M}(< A, f, s >) + \mathbf{M}(< B, f, s >)
\]

Since

\[
\int_{s \in \mathbb{R}} \mathbf{M}(< A, f, s >) \, ds \leq \text{Lip}(f) \mathbf{M}(A)
\]

and

\[
\int_{s \in \mathbb{R}} \mathbf{M}(< B, f, s >) \, ds \leq \text{Lip}(f) \mathbf{M}(B)
\]

we have,

\[
\int_{s \in \mathbb{R}} d_F^Z(< T_1, f, s >, < T_2, f, s >) \, ds \leq \text{Lip}(f) \left( \mathbf{M}(A) + \mathbf{M}(B) \right)
\]

\[
\int_{s \in \mathbb{R}} d_F^Z(< T_1, f, s >, < T_2, f, s >) \, ds \leq \text{Lip}(f) d_F^Z(T_1, T_2)
\]
Convergence of Slices

If \( d^Z_F(T_j, T_\infty) \to 0 \) and \( f : Z \to \mathbb{R} \) has \( \text{Lip}(f) \leq 1 \) then

\[
\int_{s \in \mathbb{R}} d^Z_F(<T_j, f, s>, <T_\infty, f, s>) \, ds \to 0.
\]
Convergence of Slices

If $d^Z_F(T_j, T_\infty) \to 0$ and $f : Z \to \mathbb{R}$ has $Lip(f) \leq 1$ then

$$\int_{s \in \mathbb{R}} d^Z_F(<T_j, f, s>, <T_\infty, f, s>) \, ds \to 0.$$ 

So a.e. $s \in \mathbb{R} \exists$ subseq s.t. $d^Z_F(<T_j, f, s>, <T_\infty, f, s>) \to 0$. 

[PS] also define a sliced filling volume and estimate it.
Convergence of Slices

If $d_Z^F(T_j, T_\infty) \to 0$ and $f : Z \to \mathbb{R}$ has $\text{Lip}(f) \leq 1$ then

$$\int_{s \in \mathbb{R}} d_Z^F(<T_j, f, s>, <T_\infty, f, s>) \, ds \to 0.$$ 

So a.e. $s \in \mathbb{R}$ exists subseq s.t. $d_Z^F(<T_j, f, s>, <T_\infty, f, s>) \to 0$.

What about slices of converging integral current spaces where $\text{Slice}(\langle X, d, T \rangle, f, s) = (\text{set}(<T, f, s>), d, <T, f, s>)$?
Convergence of Slices

If $d_F^Z(T_j, T_\infty) \to 0$ and $f : Z \to \mathbb{R}$ has $\text{Lip}(f) \leq 1$ then

$$\int_{s \in \mathbb{R}} d_F^Z(\langle T_j, f, s \rangle, \langle T_\infty, f, s \rangle) \, ds \to 0.$$ 

So a.e. $s \in \mathbb{R}$ exists a subsequence s.t. $d_F^Z(\langle T_j, f, s \rangle, \langle T_\infty, f, s \rangle) \to 0$.

What about slices of converging integral current spaces where $\text{Slice}((X, d, T), f, s) = (\text{set}(\langle T, f, s \rangle), d, \langle T, f, s \rangle)$?

$(X_j, d_j, T_j) \xrightarrow{\text{SWIF}} (X_\infty, d_\infty, T_\infty)$ implies

$\exists Z$ and $\varphi_j : X_j \to Z$ s.t. $d_F^Z(\varphi_j \# T_j, \varphi_\infty \# T_\infty) \to 0$. 

[PS] also define a sliced filling volume and estimate it.
Convergence of Slices

If \( d_F^{Z}(T_j, T_\infty) \to 0 \) and \( f : Z \to \mathbb{R} \) has \( \text{Lip}(f) \leq 1 \) then

\[
\int_{s \in \mathbb{R}} d_F^{Z}(< T_j, f, s >, < T_\infty, f, s >) \, ds \to 0.
\]

So a.e. \( s \in \mathbb{R} \) \( \exists \) subseq s.t. \( d_F^{Z}(< T_j, f, s >, < T_\infty, f, s >) \to 0 \).

What about slices of converging integral current spaces where \( \text{Slice}((X, d, T), f, s) = (\text{set}(< T, f, s >), d, < T, f, s >) \)?

\( (X_j, d_j, T_j) \xrightarrow{\text{SWIF}} (X_\infty, d_\infty, T_\infty) \) implies

\( \exists Z \) and \( \varphi_j : X_j \to Z \) s.t. \( d_F^{Z}(\varphi_j\# T_j, \varphi_\infty\# T_\infty) \to 0 \).

Taking \( f_j = f \circ \varphi_j \) we get subseq of sliced spaces for a.e. \( s \in \mathbb{R} \):

\( \text{Slice}(M_j, f_j, s) \xrightarrow{\text{SWIF}} \text{Slice}(M_\infty, f_\infty, s_\infty) \).
**Convergence of Slices**

If \( d_F^Z(T_j, T_\infty) \to 0 \) and \( f : Z \to \mathbb{R} \) has \( \text{Lip}(f) \leq 1 \) then

\[
\int_{s \in \mathbb{R}} d_F^Z(<T_j, f, s>, <T_\infty, f, s>) \, ds \to 0.
\]

So a.e. \( s \in \mathbb{R} \) \( \exists \) subseq s.t. \( d_F^Z(<T_j, f, s>, <T_\infty, f, s>) \to 0. \)

What about slices of converging integral current spaces where \( \text{Slice}((X, d, T), f, s) = (\text{set}(<T, f, s>), d, <T, f, s>)? \)

\((X_j, d_j, T_j) \xrightarrow{\text{SWIF}} (X_\infty, d_\infty, T_\infty) \) implies

\( \exists Z \text{ and } \varphi_j : X_j \to Z \) s.t. \( d_F^Z(\varphi_j \# T_j, \varphi_\infty \# T_\infty) \to 0. \)

Taking \( f_j = f \circ \varphi_j \) we get subseq of sliced spaces for a.e. \( s \in \mathbb{R}: \)

\( \text{Slice}(M_j, f_j, s) \xrightarrow{\text{SWIF}} \text{Slice}(M_\infty, f_\infty, s_\infty). \)

**Portegies-Sormani:** (after significant work) \( M_j \xrightarrow{\text{SWIF}} M_\infty \)

and \( p_j \in M_j \) converges to \( p_\infty \in M_\infty \) then a.e. \( s \in \mathbb{R}: \)

a subseq \( \text{Slice}(M_j, \rho_{p_j}, s) \xrightarrow{\text{SWIF}} \text{Slice}(M_\infty, \rho_{p_\infty}, s_\infty) \)
Convergence of Slices

If $d^Z_F(T_j, T_\infty) \to 0$ and $f: Z \to \mathbb{R}$ has $\text{Lip}(f) \leq 1$ then

$$\int_{s \in \mathbb{R}} d^Z_F(<T_j, f, s>, <T_\infty, f, s>) \, ds \to 0.$$ 

So a.e. $s \in \mathbb{R}$ there exists a subseq s.t. $d^Z_F(<T_j, f, s>, <T_\infty, f, s>) \to 0$.

What about slices of converging integral current spaces where $\text{Slice}((X, d, T), f, s) = (\text{set}(<T, f, s>), d, <T, f, s>)$?

$(X_j, d_j, T_j) \xrightarrow{\text{SWIF}} (X_\infty, d_\infty, T_\infty)$ implies

$\exists Z$ and $\varphi_j: X_j \to Z$ s.t. $d^Z_F(\varphi_j\# T_j, \varphi_\infty\# T_\infty) \to 0$.

Taking $f_j = f \circ \varphi_j$ we get a subseq of sliced spaces for a.e. $s \in \mathbb{R}$:

$$\text{Slice}(M_j, f_j, s) \xrightarrow{\text{SWIF}} \text{Slice}(M_\infty, f_\infty, s_\infty).$$

**Portegies-Sormani:** (after significant work) $M_j \xrightarrow{\text{SWIF}} M_\infty$ and $p_j \in M_j$ converges to $p_\infty \in M_\infty$ then a.e. $s \in \mathbb{R}$:

- a subseq $\text{Slice}(M_j, \rho_{p_j}, s) \xrightarrow{\text{SWIF}} \text{Slice}(M_\infty, \rho_{p_\infty}, s_\infty)$

So $B(p_j, s) \xrightarrow{\text{SWIF}} B(p_\infty, s)$ and $\partial B(p_j, s) \xrightarrow{\text{SWIF}} \partial B(p_\infty, s)$. 

[PS] also define a sliced filling volume and estimate it.
Convergence of Slices

If $d^Z_F(T_j, T_\infty) \to 0$ and $f : Z \to \mathbb{R}$ has $\text{Lip}(f) \leq 1$ then

$$\int_{s \in \mathbb{R}} d^Z_F(\langle T_j, f, s \rangle, \langle T_\infty, f, s \rangle) \, ds \to 0.$$ 

So a.e. $s \in \mathbb{R}$ \exists subseq s.t. $d^Z_F(\langle T_j, f, s \rangle, \langle T_\infty, f, s \rangle) \to 0$. 

What about slices of converging integral current spaces where $\text{Slice}(\langle X, d, T \rangle, f, s) = (\text{set}(\langle T, f, s \rangle), d, \langle T, f, s \rangle)$?

$(X_j, d_j, T_j) \xrightarrow{\text{SWIF}} (X_\infty, d_\infty, T_\infty)$ implies

$\exists Z$ and $\varphi_j : X_j \to Z$ s.t. $d^Z_F(\varphi_j \# T_j, \varphi_\infty \# T_\infty) \to 0$. 

Taking $f_j = f \circ \varphi_j$ we get subseq of sliced spaces for a.e. $s \in \mathbb{R}$:

$\text{Slice}(M_j, f_j, s) \xrightarrow{\text{SWIF}} \text{Slice}(M_\infty, f_\infty, s_\infty)$. 

Portegies-Sormani: (after significant work) $M_j \xrightarrow{\text{SWIF}} M_\infty$ and $p_j \in M_j$ converges to $p_\infty \in M_\infty$ then a.e. $s \in \mathbb{R}$:

a subseq $\text{Slice}(M_j, \rho_{p_j}, s) \xrightarrow{\text{SWIF}} \text{Slice}(M_\infty, \rho_{p_\infty}, s_\infty)$

So $B(p_j, s) \xrightarrow{\text{SWIF}} B(p_\infty, s)$ and $\partial B(p_j, s) \xrightarrow{\text{SWIF}} \partial B(p_\infty, s)$.

[PS] also define a sliced filling volume and estimate it.
Balls and $\mathcal{VF}$ Limits

**Portegies-Sormani:** (from last slide) $M_j \xrightarrow{\text{SWIF}} M_\infty$ and $p_j \in M_j$ converges to $p_\infty \in M_\infty$ then a.e. $s \in \mathbb{R}$:

- a subseq $\text{Slice}(M_j, \rho_{p_j}, s) \xrightarrow{\text{SWIF}} \text{Slice}(M_\infty, \rho_{p_\infty}, s_\infty)$.

So $B(p_j, s) \xrightarrow{\text{SWIF}} B(p_\infty, s)$ and $\partial B(p_j, s) \xrightarrow{\text{SWIF}} \partial B(p_\infty, s)$. 

Volume Preserving Intrinsic Flat Convergence

$M_j \xrightarrow{\text{VF}} M_\infty$ and $\lim_{j \to \infty} M_j = M_\infty$.

This implies $M_j \geq \liminf_{j \to \infty} M_j \geq \liminf_{j \to \infty} \mathcal{M}(B_j) + \liminf_{j \to \infty} \mathcal{M}(M_j \setminus B_j) \geq \mathcal{M}(B_\infty) + \mathcal{M}(M_\infty \setminus B_\infty) = \mathcal{M}(M_\infty)$.

So all are equality and so $\lim_{j \to \infty} \mathcal{M}(B(p_j, r)) = \mathcal{M}(B(p_\infty, r))$.

Portegies a la Fukaya: control eigenvalues of the spaces:

$\limsup_{j \to \infty} \lambda_k(M_j) \to \lambda_k(M_\infty)$.

Jauregui-Lee prove areas of certain surfaces converge by studying the integrals of the masses of slices.
Balls and $\mathcal{VF}$ Limits

**Portegies-Sormani:** (from last slide) $M_j \xrightarrow{\text{SWIF}} M_\infty$
and $p_j \in M_j$ converges to $p_\infty \in M_\infty$ then a.e. $s \in \mathbb{R}$:

- a subseq $\text{Slice}(M_j, \rho_{p_j}, s) \xrightarrow{\text{SWIF}} \text{Slice}(M_\infty, \rho_{p_\infty}, s_\infty)$.

So $B(p_j, s) \xrightarrow{\text{SWIF}} B(p_\infty, s)$ and $\partial B(p_j, s) \xrightarrow{\text{SWIF}} \partial B(p_\infty, s)$.

So $\liminf_{j \to \infty} M(B(p_j, s)) \geq M(B(p_\infty, s))$ and $\liminf_{j \to \infty} M(\partial B(p_j, s)) \geq M(\partial B(p_\infty, s))$. 

Portegies a la Fukaya: control eigenvalues of the spaces:

$\limsup_{j \to \infty} \lambda_k(M_j) \to \lambda_k(M_\infty)$.

Jauregui-Lee prove areas of certain surfaces converge by studying the integrals of the masses of slices.
Balls and $\mathcal{VF}$ Limits

**Portegies-Sormani:** (from last slide) $M_j \xrightarrow{\text{SWIF}} M_\infty$ and $p_j \in M_j$ converges to $p_\infty \in M_\infty$ then a.e. $s \in \mathbb{R}$:

- a subseq $\text{Slice}(M_j, \rho_{p_j}, s) \xrightarrow{\text{SWIF}} \text{Slice}(M_\infty, \rho_{p_\infty}, s_\infty)$.

So $B(p_j, s) \xrightarrow{\text{SWIF}} B(p_\infty, s)$ and $\partial B(p_j, s) \xrightarrow{\text{SWIF}} \partial B(p_\infty, s)$.

So $\liminf_{j \to \infty} M(B(p_j, s)) \geq M(B(p_\infty, s))$ and $\liminf_{j \to \infty} M(\partial B(p_j, s)) \geq M(\partial B(p_\infty, s))$.

**Volume Preserving Intrinsic Flat Convergence** $M_j \xrightarrow{\mathcal{VF}} M_\infty$:

$M_j \xrightarrow{\text{SWIF}} M_\infty$ and $\lim_{j \to \infty} M(M_j) = M(M_\infty)$.
Balls and $\mathcal{VF}$ Limits

**Portegies-Sormani:** (from last slide) $M_j \xrightarrow{\text{SWIF}} M_\infty$

and $p_j \in M_j$ converges to $p_\infty \in M_\infty$ then a.e. $s \in \mathbb{R}$:

- a subseq $\text{Slice}(M_j, \rho_{p_j}, s) \xrightarrow{\text{SWIF}} \text{Slice}(M_\infty, \rho_{p_\infty}, s_\infty)$.

So $B(p_j, s) \xrightarrow{\text{SWIF}} B(p_\infty, s)$ and $\partial B(p_j, s) \xrightarrow{\text{SWIF}} \partial B(p_\infty, s)$.

So $\lim \inf_{j \to \infty} M(B(p_j, s)) \geq M(B(p_\infty, s))$ and

$\lim \inf_{j \to \infty} M(\partial B(p_j, s)) \geq M(\partial B(p_\infty, s))$.

**Volume Preserving Intrinsic Flat Convergence** $M_j \xrightarrow{\mathcal{VF}} M_\infty$:

$M_j \xrightarrow{\text{SWIF}} M_\infty$ and $\lim_{j \to \infty} M(M_j) = M(M_\infty)$.

This implies $M(M_\infty) \geq \lim \inf_{j \to \infty} M(M_j)$.
Balls and $\mathcal{VF}$ Limits

**Portegies-Sormani:** (from last slide) $M_j \xrightarrow{\text{SWIF}} M_\infty$ and $p_j \in M_j$ converges to $p_\infty \in M_\infty$ then a.e. $s \in \mathbb{R}$:

- a subseq $\text{Slice}(M_j, \rho_{p_j}, s) \xrightarrow{\text{SWIF}} \text{Slice}(M_\infty, \rho_{p_\infty}, s_\infty)$.

So $B(p_j, s) \xrightarrow{\text{SWIF}} B(p_\infty, s)$ and $\partial B(p_j, s) \xrightarrow{\text{SWIF}} \partial B(p_\infty, s)$.

So $\lim \inf j \to \infty M(B(p_j, s)) \geq M(B(p_\infty, s))$ and $\lim \inf j \to \infty M(\partial B(p_j, s)) \geq M(\partial B(p_\infty, s))$.

**Volume Preserving Intrinsic Flat Convergence** $M_j \xrightarrow{\mathcal{VF}} M_\infty$:

- $M_j \xrightarrow{\text{SWIF}} M_\infty$ and $\lim j \to \infty M(M_j) = M(M_\infty)$.

This implies $M(M_\infty) \geq \lim \inf j \to \infty M(M_j)$

$$\geq \lim \inf j \to \infty M(B_j) + \lim \inf j \to \infty M(M_j \setminus B_j)$$
Balls and $\mathcal{VF}$ Limits

**Portegies-Sormani:** (from last slide) $M_j \xrightarrow{\text{SWIF}} M_\infty$

and $p_j \in M_j$ converges to $p_\infty \in M_\infty$ then a.e. $s \in \mathbb{R}$:

- a subseq $\text{Slice}(M_j, \rho_{p_j}, s) \xrightarrow{\text{SWIF}} \text{Slice}(M_\infty, \rho_{p_\infty}, s_\infty)$.

So $B(p_j, s) \xrightarrow{\text{SWIF}} B(p_\infty, s)$ and $\partial B(p_j, s) \xrightarrow{\text{SWIF}} \partial B(p_\infty, s)$.

So $\lim_{j \to \infty} \mathcal{M}(B(p_j, s)) \geq \mathcal{M}(B(p_\infty, s))$ and

$\lim_{j \to \infty} \mathcal{M}(\partial B(p_j, s)) \geq \mathcal{M}(\partial B(p_\infty, s))$.

**Volume Preserving Intrinsic Flat Convergence** $M_j \xrightarrow{\mathcal{VF}} M_\infty$:

$M_j \xrightarrow{\text{SWIF}} M_\infty$ and $\lim_{j \to \infty} \mathcal{M}(M_j) = \mathcal{M}(M_\infty)$.

This implies $\mathcal{M}(M_\infty) \geq \lim_{j \to \infty} \mathcal{M}(M_j)$

$\geq \lim_{j \to \infty} \mathcal{M}(B_j) + \lim_{j \to \infty} \mathcal{M}(M_j \setminus B_j)$

$\geq \mathcal{M}(B_\infty) + \mathcal{M}(M_\infty \setminus B_\infty)$
Balls and $\mathcal{VF}$ Limits

**Portegies-Sormani:** (from last slide) $M_j \xrightarrow{\text{SWIF}} M_\infty$ and $p_j \in M_j$ converges to $p_\infty \in M_\infty$ then a.e. $s \in \mathbb{R}$:

- a subseq $\text{Slice}(M_j, \rho_{p_j}, s) \xrightarrow{\text{SWIF}} \text{Slice}(M_\infty, \rho_{p_\infty}, s_\infty)$.

So $B(p_j, s) \xrightarrow{\text{SWIF}} B(p_\infty, s)$ and $\partial B(p_j, s) \xrightarrow{\text{SWIF}} \partial B(p_\infty, s)$.

So $\liminf_{j \to \infty} M(B(p_j, s)) \geq M(B(p_\infty, s))$ and

$\liminf_{j \to \infty} M(\partial B(p_j, s)) \geq M(\partial B(p_\infty, s))$.

**Volume Preserving Intrinsic Flat Convergence** $M_j \xrightarrow{\mathcal{VF}} M_\infty$:

- $M_j \xrightarrow{\text{SWIF}} M_\infty$ and $\lim_{j \to \infty} M(M_j) = M(M_\infty)$.

This implies $M(M_\infty) \geq \liminf_{j \to \infty} M(M_j)$

$\geq \liminf_{j \to \infty} M(B_j) + \liminf_{j \to \infty} M(M_j \setminus B_j)$

$\geq M(B_\infty) + M(M_\infty \setminus B_\infty) = M(M_\infty)$
Balls and $\mathcal{VF}$ Limits

**Portegies-Sormani:** (from last slide) $M_j \xrightarrow{\text{SWIF}} M_\infty$

and $p_j \in M_j$ converges to $p_\infty \in M_\infty$ then a.e. $s \in \mathbb{R}$:

a subseq $\text{Slice}(M_j, \rho_{p_j}, s) \xrightarrow{\text{SWIF}} \text{Slice}(M_\infty, \rho_{p_\infty}, s_\infty)$.

So $B(p_j, s) \xrightarrow{\text{SWIF}} B(p_\infty, s)$ and $\partial B(p_j, s) \xrightarrow{\text{SWIF}} \partial B(p_\infty, s)$.

So $\liminf_{j \to \infty} M(B(p_j, s)) \geq M(B(p_\infty, s))$ and

$\liminf_{j \to \infty} M(\partial B(p_j, s)) \geq M(\partial B(p_\infty, s))$.

**Volume Preserving Intrinsic Flat Convergence** $M_j \xrightarrow{\mathcal{VF}} M_\infty$:

$M_j \xrightarrow{\text{SWIF}} M_\infty$ and $\lim_{j \to \infty} M(M_j) = M(M_\infty)$.

This implies $M(M_\infty) \geq \liminf_{j \to \infty} M(M_j)$

$\geq \liminf_{j \to \infty} M(B_j) + \liminf_{j \to \infty} M(M_j \setminus B_j)$

$\geq M(B_\infty) + M(M_\infty \setminus B_\infty) = M(M_\infty)$

So all are equality
Balls and $\mathcal{VF}$ Limits

**Portegies-Sormani:** (from last slide) $M_j \xrightarrow{\text{SWIF}} M_\infty$
and $p_j \in M_j$ converges to $p_\infty \in M_\infty$ then a.e. $s \in \mathbb{R}$:

a subseq $\text{Slice}(M_j, \rho_{p_j}, s) \xrightarrow{\text{SWIF}} \text{Slice}(M_\infty, \rho_{p_\infty}, s_\infty)$.

So $B(p_j, s) \xrightarrow{\text{SWIF}} B(p_\infty, s)$ and $\partial B(p_j, s) \xrightarrow{\text{SWIF}} \partial B(p_\infty, s)$.

So $\liminf_{j \to \infty} M(B(p_j, s)) \geq M(B(p_\infty, s))$ and

$\liminf_{j \to \infty} M(\partial B(p_j, s)) \geq M(\partial B(p_\infty, s))$.

**Volume Preserving Intrinsic Flat Convergence** $M_j \xrightarrow{\mathcal{VF}} M_\infty$:

$M_j \xrightarrow{\text{SWIF}} M_\infty$ and $\lim_{j \to \infty} M(M_j) = M(M_\infty)$.

This implies $M(M_\infty) \geq \liminf_{j \to \infty} M(M_j)$

$\geq \liminf_{j \to \infty} M(B_j) + \liminf_{j \to \infty} M(M_j \setminus B_j)$

$\geq M(B_\infty) + M(M_\infty \setminus B_\infty) = M(M_\infty)$

So all are equality and so $\lim_{j \to \infty} M(B(p_j, r)) = M(B(p_\infty, r))$. 
Balls and $\mathcal{VF}$ Limits

**Portegies-Sormani:** (from last slide) $M_j \xrightarrow{\text{SWIF}} M_\infty$ and $p_j \in M_j$ converges to $p_\infty \in M_\infty$ then a.e. $s \in \mathbb{R}$: a subseq $\text{Slice}(M_j, \rho_{p_j}, s) \xrightarrow{\text{SWIF}} \text{Slice}(M_\infty, \rho_{p_\infty}, s_\infty)$.

So $B(p_j, s) \xrightarrow{\text{SWIF}} B(p_\infty, s)$ and $\partial B(p_j, s) \xrightarrow{\text{SWIF}} \partial B(p_\infty, s)$.

So $\liminf_{j \to \infty} M(B(p_j, s)) \geq M(B(p_\infty, s))$ and $\liminf_{j \to \infty} M(\partial B(p_j, s)) \geq M(\partial B(p_\infty, s))$.

**Volume Preserving Intrinsic Flat Convergence** $M_j \xrightarrow{\mathcal{VF}} M_\infty$:

$M_j \xrightarrow{\text{SWIF}} M_\infty$ and $\lim_{j \to \infty} M(M_j) = M(M_\infty)$.

This implies $M(M_\infty) \geq \liminf_{j \to \infty} M(M_j)$

$\geq \liminf_{j \to \infty} M(B_j) + \liminf_{j \to \infty} M(M_j \setminus B_j)$

$\geq M(B_\infty) + M(M_\infty \setminus B_\infty) = M(M_\infty)$

So all are equality and so $\lim_{j \to \infty} M(B(p_j, r)) = M(B(p_\infty, r)$.

**Portegies a la Fukaya:** control eigenvalues of the spaces:

$\limsup_{j \to \infty} \lambda_k(M_j) \to \lambda_k(M_\infty)$. 
Balls and $\mathcal{VF}$ Limits

**Portegies-Sormani:** (from last slide) $M_j \xrightarrow{\text{SWIF}} M_\infty$
and $p_j \in M_j$ converges to $p_\infty \in M_\infty$ then a.e. $s \in \mathbb{R}$:

a subseq $\text{Slice}(M_j, \rho_{p_j}, s) \xrightarrow{\text{SWIF}} \text{Slice}(M_\infty, \rho_{p_\infty}, s_\infty)$.

So $B(p_j, s) \xrightarrow{\text{SWIF}} B(p_\infty, s)$ and $\partial B(p_j, s) \xrightarrow{\text{SWIF}} \partial B(p_\infty, s)$.

So $\lim \inf_{j \to \infty} M(B(p_j, s)) \geq M(B(p_\infty, s))$ and

$\lim \inf_{j \to \infty} M(\partial B(p_j, s)) \geq M(\partial B(p_\infty, s))$.

**Volume Preserving Intrinsic Flat Convergence** $M_j \xrightarrow{\mathcal{VF}} M_\infty$:

$M_j \xrightarrow{\text{SWIF}} M_\infty$ and $\lim_{j \to \infty} M(M_j) = M(M_\infty)$.

This implies $M(M_\infty) \geq \lim \inf_{j \to \infty} M(M_j)$

$\geq \lim \inf_{j \to \infty} M(B_j) + \lim \inf_{j \to \infty} M(M_j \setminus B_j)$

$\geq M(B_\infty) + M(M_\infty \setminus B_\infty) = M(M_\infty)$

So all are equality and so $\lim_{j \to \infty} M(B(p_j, r)) = M(B(p_\infty, r)$.

**Portegies a la Fukaya:** control eigenvalues of the spaces:

$\lim \sup_{j \to \infty} \lambda_k(M_j) \to \lambda_k(M_\infty)$.

**Jauregui-Lee** prove areas of certain surfaces converge
by studying the integrals of the masses of slices.
IAS Emerging Topic Conjecture [Gromov-S]

Suppose $M^3_j$ have $\text{Vol}(M^3_j) \leq V$ and $\text{Diam}(M^3_j) \leq D$
IAS Emerging Topic Conjecture [Gromov-S]

Suppose $M_j^3$ have $\text{Vol}(M_j^3) \leq V$ and $\text{Diam}(M_j^3) \leq D$

by [Wenger]: subseq $M_j \xrightarrow{\text{SWIF}} M_\infty$ possibly 0.

Conjecture: If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$

where $\text{MinA}_j = \min \{ \text{Area}(\Sigma) : \text{closed min surfaces} \Sigma \subset M_j^3 \}$

Then $M_\infty$ has generalized "Scalar $\geq 0"$

Furthermore: we believe that we have $M_j \xrightarrow{\text{VF}} M_\infty$

where $M_\infty$ is a connected length space with Euclidean tangent cones.

How can we show $M_\infty$ is connected?

Does it contain geodesics?

For GH limits, Gromov proved midpoints converged to midpoints, but for SWIF limits midpoints might disappear....

Given $p, q \in M_\infty$, $\exists p_j, q_j \in M_j$ converging to $p, q$.

Taking $x_j$ to be a midpoint between $p_j$ and $q_j$, can we show $\text{FillVol}(B(x_j, r)) \geq C V, D, A r^3$?

OPEN.

We must use $x_j$ a midpoint because other points can disappear.

Perhaps use sliced filling volumes with $f_j(\cdot) = d_j(\cdot, p_j)$?

OPEN.
IAS Emerging Topic Conjecture [Gromov-S]

**Suppose** $M_j^3$ have $\text{Vol}(M_j^3) \leq V$ and $\text{Diam}(M_j^3) \leq D$

by [Wenger]: subseq $M_j \xrightarrow{\text{SWIF}} M_\infty$ possibly 0.

and by [SW]: $\liminf_{j \to \infty} \text{Vol}(M_j) \geq \mathbf{M}(M_\infty)$.

Conjecture: If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$ where $\text{MinA}_j = \min \{ \text{Area}(\Sigma) : \text{closed min surfaces} \Sigma \subset M_j^3 \}$

Then $M_\infty$ has generalized "Scalar $\geq 0"$.

Furthermore: we believe that we have $M_j \xrightarrow{\text{VF}} M_\infty$ where $M_\infty$ is a connected length space with Euclidean tangent cones.

How can we show $M_\infty$ is connected? Does it contain geodesics?

For GH limits, Gromov proved midpoints converged to midpoints, but for SWIF limits midpoints might disappear.

Given $p, q \in M_\infty$, $\exists p_j, q_j \in M_j$ converging to $p, q$.

Taking $x_j$ to be a midpoint between $p_j$ and $q_j$, can we show $\text{FillVol}(B(x_j, r)) \geq C V, D, A r^3$?

OPEN. We must use $x_j$ a midpoint because other points can disappear. Perhaps use sliced filling volumes with $\text{f}_j(\cdot) = d_j(\cdot, p_j)$?

OPEN.
IAS Emerging Topic Conjecture [Gromov-S]

**Suppose** $M_j^3$ have $\text{Vol}(M_j^3) \leq V$ and $\text{Diam}(M_j^3) \leq D$

by [Wenger]: subseq $M_j \xrightarrow{\text{SWIF}} M_\infty$ possibly 0.

and by [SW]: $\liminf_{j \to \infty} \text{Vol}(M_j) \geq \text{M}(M_\infty)$.

and by [SW]: If $M_\infty \neq 0$ then it is m-rectifiable.
IAS Emerging Topic Conjecture [Gromov-S]

Suppose $M_j^3$ have $\text{Vol}(M_j^3) \leq V$ and $\text{Diam}(M_j^3) \leq D$

by [Wenger]: subseq $M_j \xrightarrow{\text{SWIF}} M_\infty$ possibly 0.

and by [SW]: $\lim \inf_{j \to \infty} \text{Vol}(M_j) \geq \mathcal{M}(M_\infty)$.

and by [SW]: If $M_\infty \neq 0$ then it is m-rectifiable.

Conjecture: If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$

OPEN.

We must use $x_j$ as a midpoint because other points can disappear.

Perhaps use sliced filling volumes with $f_j(\cdot) = d_j(\cdot, p_j)$?
IAS Emerging Topic Conjecture [Gromov-S]

Suppose $M_j^3$ have $\text{Vol}(M_j^3) \leq V$ and $\text{Diam}(M_j^3) \leq D$

by [Wenger]: subseq $M_j \xrightarrow{\text{SWIF}} M_\infty$ possibly 0.
and by [SW]: $\liminf_{j \to \infty} \text{Vol}(M_j) \geq \mathcal{M}(M_\infty)$.
and by [SW]: If $M_\infty \neq 0$ then it is m-rectifiable.

Conjecture: If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$
where $\text{MinA}_j = \min\{\text{Area}(\Sigma) : \text{closed min surfaces } \Sigma \subset M_j^3\}$
IAS Emerging Topic Conjecture [Gromov-S]

**Suppose** $M_j^3$ have $\text{Vol}(M_j^3) \leq V$ and $\text{Diam}(M_j^3) \leq D$

by [Wenger]: subseq $M_j \xrightarrow{\text{SWIF}} M_\infty$ possibly 0.
and by [SW]: $\liminf_{j \to \infty} \text{Vol}(M_j) \geq \mathcal{M}(M_\infty)$.
and by [SW]: If $M_\infty \neq 0$ then it is m-rectifiable.

**Conjecture:** If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$
where $\text{MinA}_j = \min\{\text{Area}(\Sigma) : \text{closed min surfaces } \Sigma \subset M_j^3\}$

Then $M_\infty$ has generalized “$\text{Scalar} \geq 0$”
IAS Emerging Topic Conjecture [Gromov-S]

**Suppose** $M^3_j$ have $\text{Vol}(M^3_j) \leq V$ and $\text{Diam}(M^3_j) \leq D$

by [Wenger]: subseq $M_j \xrightarrow{\text{SWIF}} M_\infty$ possibly 0.

and by [SW]: $\lim \inf_{j \to \infty} \text{Vol}(M_j) \geq \mathcal{M}(M_\infty)$.

and by [SW]: If $M_\infty \neq 0$ then it is m-rectifiable.

**Conjecture:** If in addition we have $\text{Scalar}_j \geq 0$ and $\text{Min}A_j \geq A$

where $\text{Min}A_j = \min\{\text{Area}(\Sigma) : \text{closed min surfaces } \Sigma \subset M^3_j\}$

Then $M_\infty$ has generalized “Scalar $\geq 0$”

**Furthermore:** we believe that we have $M_j \xrightarrow{\nu_\mathcal{F}} M_\infty$ where $M_\infty$ is a connected length space with Euclidean tangent cones.
IAS Emerging Topic Conjecture [Gromov-S]

Suppose $M_j^3$ have $\text{Vol}(M_j^3) \leq V$ and $\text{Diam}(M_j^3) \leq D$

by [Wenger]: subseq $M_j \xrightarrow{\text{SWIF}} M_\infty$ possibly 0.

and by [SW]: $\lim \inf_{j \to \infty} \text{Vol}(M_j) \geq \mathbf{M}(M_\infty)$.

and by [SW]: If $M_\infty \neq 0$ then it is m-rectifiable.

Conjecture: If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$
where $\text{MinA}_j = \min\{\text{Area}(\Sigma) : \text{closed min surfaces } \Sigma \subset M_j^3\}$

Then $M_\infty$ has generalized “Scalar $\geq 0$”

Furthermore: we believe that we have $M_j \xrightarrow{\nu, \mathcal{F}} M_\infty$ where $M_\infty$ is a connected length space with Euclidean tangent cones.
Suppose \( M_j^3 \) have \( \text{Vol}(M_j^3) \leq V \) and \( \text{Diam}(M_j^3) \leq D \)

by [Wenger]: subseq \( M_j \overset{\text{SWIF}}{\longrightarrow} M_\infty \) possibly 0.

and by [SW]: \( \lim_{j \to \infty} \text{Vol}(M_j) \geq M(M_\infty) \).

and by [SW]: If \( M_\infty \neq 0 \) then it is \( m \)-rectifiable.

**Conjecture:** If in addition we have \( \text{Scalar}_j \geq 0 \) and \( \text{MinA}_j \geq A \)

where \( \text{MinA}_j = \min\{\text{Area}(\Sigma) : \text{closed min surfaces } \Sigma \subset M_j^3\} \)

Then \( M_\infty \) has generalized “\( \text{Scalar} \geq 0 \)”

**Furthermore:** we believe that we have \( M_j \overset{\nu,F}{\longrightarrow} M_\infty \) where \( M_\infty \) is a connected length space with Euclidean tangent cones.

How can we show \( M_\infty \) is connected?
IAS Emerging Topic Conjecture [Gromov-S]

Suppose $M_j^3$ have $\text{Vol}(M_j^3) \leq V$ and $\text{Diam}(M_j^3) \leq D$

by [Wenger]: subseq $M_j \xrightarrow{\text{SWIF}} M_\infty$ possibly 0.
and by [SW]: $\lim \inf_{j \to \infty} \text{Vol}(M_j) \geq \mathcal{M}(M_\infty)$.
and by [SW]: If $M_\infty \neq 0$ then it is m-rectifiable.

**Conjecture:** If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$ where $\text{MinA}_j = \min\{\text{Area}(\Sigma) : \text{closed min surfaces } \Sigma \subset M_j^3\}$

Then $M_\infty$ has generalized “$\text{Scalar} \geq 0$”

**Furthermore:** we believe that we have $M_j \xrightarrow{\nuF} M_\infty$ where $M_\infty$ is a connected length space with Euclidean tangent cones.

How can we show $M_\infty$ is connected? Does it contain geodesics?
IAS Emerging Topic Conjecture [Gromov-S]

Suppose $M_j^3$ have $\text{Vol}(M_j^3) \leq V$ and $\text{Diam}(M_j^3) \leq D$

by [Wenger]: subseq $M_j \xrightarrow{\text{SWIF}} M_\infty$ possibly 0.

and by [SW]: $\liminf_{j \to \infty} \text{Vol}(M_j) \geq \mathbf{M}(M_\infty)$.

and by [SW]: If $M_\infty \neq 0$ then it is m-rectifiable.

Conjecture: If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$
where $\text{MinA}_j = \min\{\text{Area}(\Sigma) : \text{closed min surfaces } \Sigma \subset M_j^3\}$

Then $M_\infty$ has generalized “Scalar $\geq 0$”

Furthermore: we believe that we have $M_j \xrightarrow{\nu\mathcal{F}} M_\infty$ where
$M_\infty$ is a connected length space with Euclidean tangent cones.

How can we show $M_\infty$ is connected? Does it contain geodesics?
For GH limits, Gromov proved midpoints converged to midpoints,
IAS Emerging Topic Conjecture [Gromov-S]

Suppose $M^3_j$ have $\text{Vol}(M^3_j) \leq V$ and $\text{Diam}(M^3_j) \leq D$

by [Wenger]: subseq $M_j \xrightarrow{\text{SWIF}} M_\infty$ possibly 0.
and by [SW]: $\lim inf_{j \to \infty} \text{Vol}(M_j) \geq \mathcal{M}(M_\infty)$.
and by [SW]: If $M_\infty \neq 0$ then it is $m$-rectifiable.

Conjecture: If in addition we have $\textbf{Scalar}_j \geq 0$ and $\textbf{MinA}_j \geq A$
where $\textbf{MinA}_j = \min \{ \text{Area}(\Sigma) : \text{closed min surfaces } \Sigma \subset M^3_j \}$
Then $M_\infty$ has generalized “Scalar $\geq 0$”

Furthermore: we believe that we have $M_j \xrightarrow{\nu,F} M_\infty$ where
$M_\infty$ is a connected length space with Euclidean tangent cones.
How can we show $M_\infty$ is connected? Does it contain geodesics?
For GH limits, Gromov proved midpoints converged to midpoints, but for SWIF limits midpoints might disappear....
IAS Emerging Topic Conjecture [Gromov-S]

Suppose $M_j^3$ have $\text{Vol}(M_j^3) \leq V$ and $\text{Diam}(M_j^3) \leq D$

by [Wenger]: subseq $M_j \xrightarrow{\text{SWIF}} M_\infty$ possibly 0.
and by [SW]: $\lim \inf_{j \to \infty} \text{Vol}(M_j) \geq \mathcal{M}(M_\infty)$.
and by [SW]: If $M_\infty \neq 0$ then it is m-rectifiable.

Conjecture: If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$
where $\text{MinA}_j = \min \{\text{Area}(\Sigma) : \text{closed min surfaces } \Sigma \subset M_j^3\}$

Then $M_\infty$ has generalized “Scalar $\geq 0”$

Furthermore: we believe that we have $M_j \xrightarrow{\nu\mathcal{F}} M_\infty$ where
$M_\infty$ is a connected length space with Euclidean tangent cones.

How can we show $M_\infty$ is connected? Does it contain geodesics?
For GH limits, Gromov proved midpoints converged to midpoints, but for SWIF limits midpoints might disappear....

Given $p, q \in M_\infty$, $\exists p_j, q_j \in M_j$ converging to $p, q$.
Taking $x_j$ to be a midpoint between $p_j$ and $q_j$,..
IAS Emerging Topic Conjecture [Gromov-S]

Suppose $M_j^3$ have $\text{Vol}(M_j^3) \leq V$ and $\text{Diam}(M_j^3) \leq D$

by [Wenger]: subseq $M_j \xrightarrow{\text{SWIF}} M_\infty$ possibly 0.
and by [SW]: $\lim \inf_{j \to \infty} \text{Vol}(M_j) \geq M(M_\infty)$.
and by [SW]: If $M_\infty \neq 0$ then it is m-rectifiable.

Conjecture: If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$
where $\text{MinA}_j = \min \{ \text{Area}(\Sigma) : \text{closed min surfaces } \Sigma \subset M_j^3 \}$
Then $M_\infty$ has generalized “$\text{Scalar} \geq 0$”

Furthermore: we believe that we have $M_j \xrightarrow{\nu F} M_\infty$ where
$M_\infty$ is a connected length space with Euclidean tangent cones.
How can we show $M_\infty$ is connected? Does it contain geodesics?
For GH limits, Gromov proved midpoints converged to midpoints,
but for SWIF limits midpoints might disappear....
Given $p, q \in M_\infty$, $\exists p_j, q_j \in M_j$ converging to $p, q$.
Taking $x_j$ to be a midpoint between $p_j$ and $q_j$,
can we show $\text{FillVol}(B(x_j, r)) \geq C_{V,D,A}r^3$?
IAS Emerging Topic Conjecture [Gromov-S]

Suppose $M^3_j$ have $\text{Vol}(M^3_j) \leq V$ and $\text{Diam}(M^3_j) \leq D$

by [Wenger]: subseq $M_j \xrightarrow{\text{SWIF}} M_\infty$ possibly 0.
and by [SW]: $\lim \inf_{j \to \infty} \text{Vol}(M_j) \geq M(M_\infty)$.
and by [SW]: If $M_\infty \neq 0$ then it is m-rectifiable.

Conjecture: If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$
where $\text{MinA}_j = \min\{\text{Area}(\Sigma) : \text{closed min surfaces } \Sigma \subset M^3_j\}$
Then $M_\infty$ has generalized “Scalar $\geq 0$”

Furthermore: we believe that we have $M_j \xrightarrow{\nu,F} M_\infty$ where
$M_\infty$ is a connected length space with Euclidean tangent cones.
How can we show $M_\infty$ is connected? Does it contain geodesics?
For GH limits, Gromov proved midpoints converged to midpoints, but for SWIF limits midpoints might disappear....

Given $p, q \in M_\infty$, $\exists p_j, q_j \in M_j$ converging to $p, q$.
Taking $x_j$ to be a midpoint between $p_j$ and $q_j$,
can we show $\text{FillVol}(B(x_j, r)) \geq C_{V,D,A} r^3$? OPEN.
We must use $x_j$ a midpoint because other points can disappear.
IAS Emerging Topic Conjecture [Gromov-S]

Suppose $M_j^3$ have $\text{Vol}(M_j^3) \leq V$ and $\text{Diam}(M_j^3) \leq D$

by [Wenger]: subseq $M_j \xrightarrow{\text{SWIF}} M_\infty$ possibly 0.

and by [SW]: $\lim \inf_{j \to \infty} \text{Vol}(M_j) \geq \mathcal{M}(M_\infty)$.

and by [SW]: If $M_\infty \neq 0$ then it is m-rectifiable.

**Conjecture:** If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$

where $\text{MinA}_j = \min \{ \text{Area}(\Sigma) : \text{closed min surfaces } \Sigma \subset M_j^3 \}$

Then $M_\infty$ has generalized “Scalar $\geq 0”$

**Furthermore:** we believe that we have $M_j \xrightarrow{\nu_{\mathcal{F}}} M_\infty$ where

$M_\infty$ is a connected length space with Euclidean tangent cones.

How can we show $M_\infty$ is connected? Does it contain geodesics?

For GH limits, Gromov proved midpoints converged to midpoints, but for SWIF limits midpoints might disappear....

Given $p, q \in M_\infty$, $\exists p_j, q_j \in M_j$ converging to $p, q$.

Taking $x_j$ to be a midpoint between $p_j$ and $q_j$,

can we show $\text{FillVol}(B(x_j, r)) \geq C_{V,D,A}r^3$? OPEN.

We must use $x_j$ a midpoint because other points can disappear.

Perhaps use sliced filling volumes with $f_j(\cdot) = d_j(\cdot, p_j)$?
IAS Emerging Topic Conjecture [Gromov-S]

Suppose $M^3_j$ have $\text{Vol}(M^3_j) \leq V$ and $\text{Diam}(M^3_j) \leq D$

by [Wenger]: subseq $M_j \xrightarrow{\text{SWIF}} M_\infty$ possibly 0.

and by [SW]: $\lim \inf_{j \to \infty} \text{Vol}(M_j) \geq M(M_\infty)$.

and by [SW]: If $M_\infty \neq 0$ then it is m-rectifiable.

**Conjecture:** If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$ where $\text{MinA}_j = \min \{ \text{Area}(\Sigma) : \text{closed min surfaces } \Sigma \subset M^3_j \}$

Then $M_\infty$ has generalized “$\text{Scalar} \geq 0$”

**Furthermore:** we believe that we have $M_j \xrightarrow{\nu,F} M_\infty$ where $M_\infty$ is a connected length space with Euclidean tangent cones.

How can we show $M_\infty$ is connected? Does it contain geodesics?

For GH limits, Gromov proved midpoints converged to midpoints, but for SWIF limits midpoints might disappear....

Given $p, q \in M_\infty$, $\exists p_j, q_j \in M_j$ converging to $p, q$.

Taking $x_j$ to be a midpoint between $p_j$ and $q_j$,

can we show $\text{FillVol}(B(x_j, r)) \geq C_{V,D,A} r^3$? OPEN.

We must use $x_j$ a midpoint because other points can disappear.

Perhaps use sliced filling volumes with $f_j(\cdot) = d_j(\cdot, p_j)$? OPEN
IAS Emerging Topic Conjecture: Tan Cones

Suppose $M_j^3$ have $\text{Vol}(M_j^3) \leq V$ and $\text{Diam}(M_j^3) \leq D$

So a subsequence $M_j \overset{\text{SWIF}}{\to} M_\infty$.

**Conjecture**: If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$ where

$$\text{MinA}_j = \min\{\text{Area}(\Sigma) : \text{closed min surfaces } \Sigma \subset M_j^3\}$$

Then $M_\infty$ has generalized “Scalar $\geq 0$”

**Furthermore**: we believe that we have $M_j \overset{\nu_{\mathcal{F}}}{\to} M_\infty$ where $M_\infty$ is a connected length space with Euclidean tangent cones.
IAS Emerging Topic Conjecture: Tan Cones

Suppose $M_j^3$ have $\text{Vol}(M_j^3) \leq V$ and $\text{Diam}(M_j^3) \leq D$

So a subsequence $M_j \xrightarrow{SWIF} M_\infty$.

**Conjecture:** If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$ where $\text{MinA}_j = \min \{ \text{Area}(\Sigma) : \text{closed min surfaces } \Sigma \subset M_j^3 \}$

Then $M_\infty$ has generalized “Scalar $\geq 0$”

**Furthermore:** we believe that we have $M_j \xrightarrow{V,F} M_\infty$ where $M_\infty$ is a connected length space with Euclidean tangent cones.
IAS Emerging Topic Conjecture: Tan Cones

Suppose $M^3_j$ have $\text{Vol}(M^3_j) \leq V$ and $\text{Diam}(M^3_j) \leq D$

So a subsequence $M_j \xrightarrow{\text{SWIF}} M_{\infty}$.

Conjecture: If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$ where $\text{MinA}_j = \min\{\text{Area}(\Sigma) : \text{closed min surfaces } \Sigma \subset M^3_j\}$

Then $M_{\infty}$ has generalized “Scalar $\geq 0$”

Furthermore: we believe that we have $M_j \xrightarrow{\nu_F} M_{\infty}$ where $M_{\infty}$ is a connected length space with Euclidean tangent cones. How would we prove the tangent cones are Euclidean?
IAS Emerging Topic Conjecture: Tan Cones

**Suppose** $M_j^3$ have $\text{Vol}(M_j^3) \leq V$ and $\text{Diam}(M_j^3) \leq D$

So a subsequence $M_j \xrightarrow{\text{SWIF}} M_\infty$.

**Conjecture:** If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$ where $\text{MinA}_j = \min \{\text{Area}(\Sigma) : \text{closed min surfaces } \Sigma \subset M_j^3\}$

Then $M_\infty$ has generalized “Scalar $\geq 0$”

**Furthermore:** we believe that we have $M_j \xrightarrow{VF} M_\infty$ where $M_\infty$ is a connected length space with Euclidean tangent cones.

How would we prove the tangent cones are Euclidean?

By Ambrosio-Kirchheim theory, we know that at a.e. $p \in M_\infty$, there is a tangent cone, $T_pM$, which is a normed vector space:

$$(B(p, r_i), d/r_i, [[B(p, r_i)]] \xrightarrow{\text{SWIF}} B(0, 1) \subset T_pM).$$
IAS Emerging Topic Conjecture: Tan Cones

Suppose $M_j^3$ have $\text{Vol}(M_j^3) \leq V$ and $\text{Diam}(M_j^3) \leq D$

So a subsequence $M_j \xrightarrow{\text{SWIF}} M_\infty$.

**Conjecture:** If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$ where $\text{MinA}_j = \min\{\text{Area}(\Sigma) : \text{closed min surfaces } \Sigma \subset M_j^3\}$

Then $M_\infty$ has generalized “Scalar $\geq 0$”

**Furthermore:** we believe that we have $M_j \xrightarrow{\mathcal{V}} M_\infty$ where $M_\infty$ is a connected length space with Euclidean tangent cones.

How would we prove the tangent cones are Euclidean?

By Ambrosio-Kirchheim theory, we know that at a.e. $p \in M_\infty$, there is a tangent cone, $T_p M$, which is a normed vector space:

$$(B(p, r_i), d/r_i, [[B(p, r_i)]] \xrightarrow{\text{SWIF}} B(0, 1) \subset T_p M).$$

So perhaps we could use geometric stability of a rigidity theorem that implies a ball is a Euclidean ball to prove this.
IAS Emerging Topic Conjecture: Tan Cones

Suppose $M_j^3$ have $\text{Vol}(M_j^3) \leq V$ and $\text{Diam}(M_j^3) \leq D$

So a subsequence $M_j \xrightarrow{\text{SWIF}} M_\infty$.

**Conjecture:** If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$ where $\text{MinA}_j = \min \{\text{Area}(\Sigma) : \text{closed min surfaces } \Sigma \subset M_j^3\}$

Then $M_\infty$ has generalized “$\text{Scalar} \geq 0$”

**Furthermore:** we believe that we have $M_j \xrightarrow{\mathcal{V}} M_\infty$ where $M_\infty$ is a connected length space with Euclidean tangent cones.

How would we prove the tangent cones are Euclidean?

By Ambrosio-Kirchheim theory, we know that at a.e. $p \in M_\infty$, there is a tangent cone, $T_p M$, which is a normed vector space:

$$(B(p, r_i), d/r_i, [[B(p, r_i)]]) \xrightarrow{\text{SWIF}} B(0, 1) \subset T_p M.$$  

So perhaps we could use geometric stability of a rigidity theorem that implies a ball is a Euclidean ball to prove this.

Note that $\lambda(p) = 1$ if $T_p M$ is Euclidean, so $\| T_\infty \| = 1 \cdot \theta \cdot H^3$. 

IAS Emerging Topic Conjecture: Tan Cones

Suppose $M^3_j$ have $\text{Vol}(M^3_j) \leq V$ and $\text{Diam}(M^3_j) \leq D$

So a subsequence $M_j \xrightarrow{\text{SWIF}} M_\infty$.

**Conjecture:** If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$ where $\text{MinA}_j = \min\{\text{Area}(\Sigma) : \text{closed min surfaces } \Sigma \subset M^3_j\}$

Then $M_\infty$ has generalized “Scalar $\geq 0$”

Furthermore: we believe that we have $M_j \xrightarrow{\mathcal{V}_\mathcal{F}} M_\infty$ where $M_\infty$ is a connected length space with Euclidean tangent cones.

How would we prove the tangent cones are Euclidean?

By Ambrosio-Kirchheim theory, we know that at a.e. $p \in M_\infty$, there is a tangent cone, $T_p M$, which is a normed vector space:

$$(B(p, r_i), d/r_i, [[B(p, r_i)]])) \xrightarrow{\text{SWIF}} B(0, 1) \subset T_p M.$$ 

So perhaps we could use geometric stability of a rigidity theorem that implies a ball is a Euclidean ball to prove this.

Note that $\lambda(p) = 1$ if $T_p M$ is Euclidean, so $||T_\infty|| = 1 \cdot \theta \cdot \mathbb{H}^3$.

**Conjecture:** the weight $\theta = 1$. 
IAS Emerging Topic Conjecture: Tan Cones

Suppose $M^3_j$ have $\text{Vol}(M^3_j) \leq V$ and $\text{Diam}(M^3_j) \leq D$

So a subsequence $M_j^{\text{SWIF}} \rightarrow M_{\infty}$.

Conjecture: If in addition we have $\text{Scalar}_j \geq 0$ and $\text{MinA}_j \geq A$ where $\text{MinA}_j = \min\{\text{Area}(\Sigma) : \text{closed min surfaces } \Sigma \subset M^3_j\}$

Then $M_{\infty}$ has generalized “Scalar $\geq 0$”

Furthermore: we believe that we have $M_j^{\nu_F} \rightarrow M_{\infty}$ where $M_{\infty}$ is a connected length space with Euclidean tangent cones.

How would we prove the tangent cones are Euclidean?

By Ambrosio-Kirchheim theory, we know that at a.e. $p \in M_{\infty}$, there is a tangent cone, $T_p M$, which is a normed vector space:

$$(B(p, r_i), d/r_i, [[B(p, r_i)]] )^{\text{SWIF}} \rightarrow B(0, 1) \subset T_p M.$$ 

So perhaps we could use geometric stability of a rigidity theorem that implies a ball is a Euclidean ball to prove this.

Note that $\lambda(p) = 1$ if $T_p M$ is Euclidean, so $||T_{\infty}|| = 1 \cdot \theta \cdot \mathbb{H}^3$.

Conjecture: the weight $\theta = 1$. So $||T_{\infty}|| = \mathbb{H}^3$.

Open: Prove $||T_{\infty}|| = \mathbb{H}^3$. (Ricci case by Colding “Volumes....”).