Intrinsic Flat and Gromov-Hausdorff Convergence

Christina Sormani

CUNY GC and Lehman College

Lectures III-IV: Proving Intrinsic Flat Convergence

Intrinsic Flat and Gromov-Hausdorff Convergence

Lecture 1: Geometric Notions of Convergence DONE!

Reviewed C^k, C^0, Lip, and GH Convergence,
Sormani-Wenger Intrinsic Flat Convergence (SWIF) or ($\mathcal{F}$),
Volume Preserving Intrinsic Flat Convergence ($\mathcal{VF}$)
Allen-Perales-Sormani (VADB) Convergence

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Lecture 2: Open Problems on Scalar Curvature DONE!

Consider: Three Dimensional Manifolds M_j^3 with $Scal \geq H$
and their Limit Spaces M_∞
Which Geometric Properties of M_j^3 with $Scal \geq H$
persist on their Limit Spaces M_∞?
Which Rigidity Theorems for M^3 with $Scal \geq H$
have Geometric Stability?
[Conjectures on Conv and Scalar (2021) arXiv: 2103.10093]

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Lectures 3&4: Techniques to Apply to Prove Convergence

See https://sites.google.com/site/intrinsicflatconvergence/

Volume Preserving Intrinsic Flat $\mathcal{VF}$ Convergence

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Defn: $M_j \xrightarrow{\mathcal{F}} M_\infty$ if $d_{\mathcal{F}}(M_j, M_\infty) = d_{SWIF}(M_j, M_\infty) \to 0$:

Sormani-Wenger: Intrinsic Flat Distance

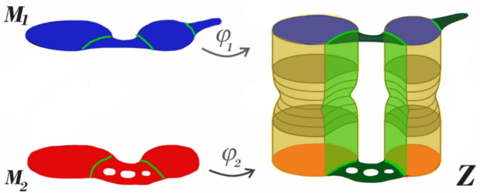
The *intrinsic flat distance* between oriented manifolds M_i^m is:

$$d_{SWIF}(M_1^m, M_2^m) = \inf\left\{ d_F^Z\left(\varphi_{1\#}[[M_1^m]], \varphi_{2\#}[[M_2^m]]\right) \mid \varphi_i : M_i^m \to Z \right\}$$

where the infimum is taken over all complete metric spaces, Z, and over all distance preserving maps $\varphi_i : M_i^m \to Z$.

Here: $d_F^Z\left(\varphi_{1\#}[[M_1^m]], \varphi_{2\#}[[M_2^m]]\right)$ is the Federer-Fleming Flat dist

$$= \inf\left\{ \underset{area}{\mathbf{M}}(A) + \underset{vol}{\mathbf{M}}(B) : A + \partial B = \varphi_{1\#}[[M_1^m]] - \varphi_{2\#}[[M_2^m]] \right\}$$



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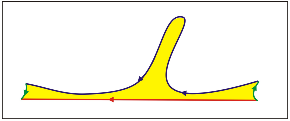
Whitney (1957): Flat norm between submanifolds N_i in $\mathbb{R}^N$:

$$|N_1 - N_2|_\flat = \inf \left\{ \mathbf{M}(A) + \mathbf{M}(B) \right\}$$

where A and B are **chains**

such that $A + \partial B = N_1 - N_2$.



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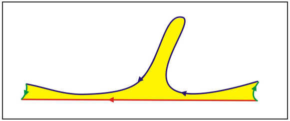

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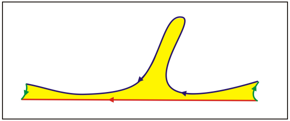

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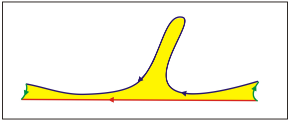

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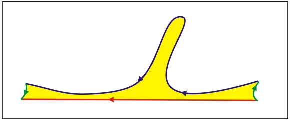

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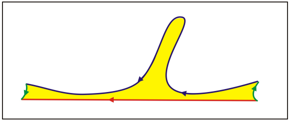

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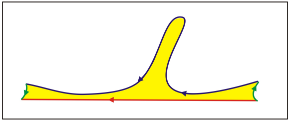

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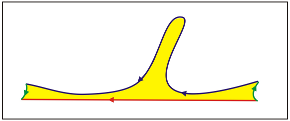

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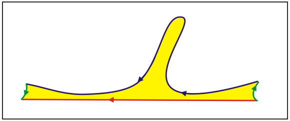

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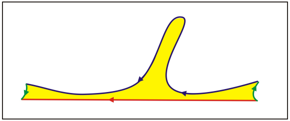

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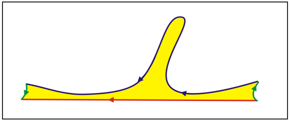

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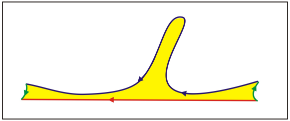

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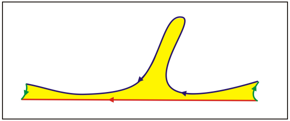

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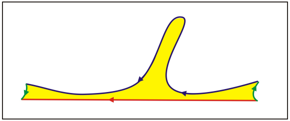

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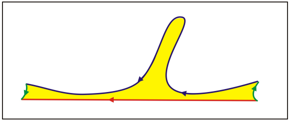

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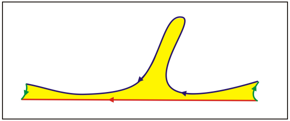

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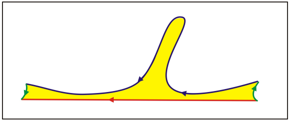

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Currents in Metric Spaces:

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$$\operatorname{set}(T) = \{z \in Z \,|\, \liminf_{r \to 0} ||T||(B(z, r))/r^m > 0\}.$$

Note: $\operatorname{set}(T)$ is cntbly rectifiable: $\mathcal{H}^m\left(\operatorname{set}(T) \setminus \bigcup_{i=1}^{\infty} \varphi_i(A_i)\right) = 0$.

Compactness Thm [AK]: If integral currents T_j have $\operatorname{set}(T_j) \subset K$ compact, and $\mathbf{M}(T_j) \le V$ and $\mathbf{M}(\partial T_j) \le A$ then $\exists$ subseq T_{j_k} and an integral current T_∞ s.t. $T_j(\omega) \to T_\infty(\omega)\, \forall \omega$

Ambrosio-Kirchheim Compactness Theorem:

Ambrosio-Kirchheim (2000): an integral current, T, is an integer rectifiable current s.t. ∂T is also integer rectifiable. where an integer rectifiable current, T, has cntbly many pairwise disjoint biLip charts $\varphi_i : A_i \to \varphi_i(A_i) \subset Z$ and weights $a_i \in \mathbb{Z}$ s.t.

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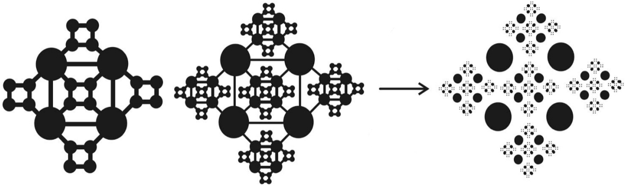

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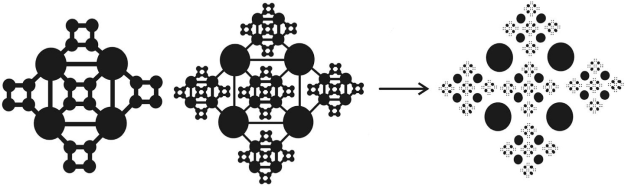

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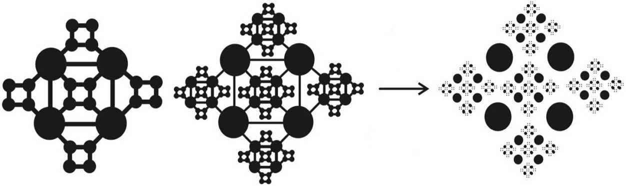

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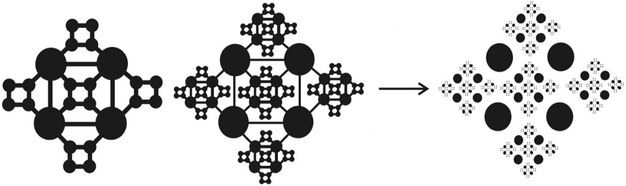

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Now we can truly define integral current spaces.

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Thus X is cntbly $\mathcal{H}^m$ rectifiable: it has cntbly many pairwise disjoint Lip charts $\varphi_i : A_i \to X$ s.t. $\mathcal{H}^m(X \setminus \bigcup_{i=1}^{\infty} \varphi_i(A_i)) = 0$.

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Sormani-Wenger: Intrinsic Flat Distance

between integral current spaces $M_i^m = (X_i, d_i, T_i)$ is:

$$d_{SWIF}(M_1^m, M_2^m) = \inf \left\{ d_F^Z \left(\varphi_{1\#} T_1, \varphi_{2\#} T_2 \right) \mid \varphi_i : M_i^m \to Z \right\}$$

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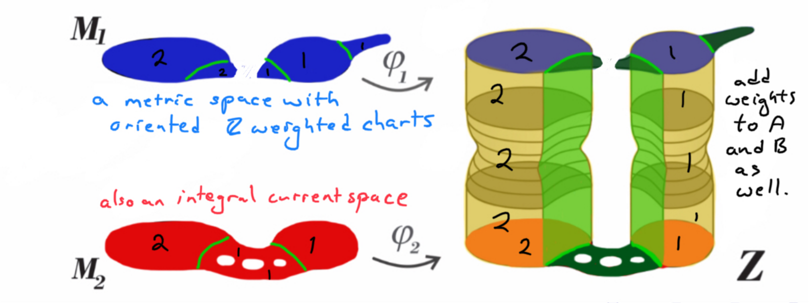

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Thm [SW-JDG]: If M_i are compact and $d_{SWIF}(M_1, M_2) = 0$ then $\exists$ a current preserving isometry $F : M_1 \to M_2$: $d_2(F(p), F(q)) = d_1(p, q)\ \forall p, q \in X_1$ and $F_\# T_1 = T_2$.

Pf: Show inf achieved: $\exists \varphi_i : M_i \to Z$ s.t. $\varphi_{1\#} T_1 = \varphi_{2\#} T_2$.
So $\mathrm{set}(\varphi_{1\#} T_1) = \mathrm{set}(\varphi_{2\#} T_2)$ and $F = \varphi_2^{-1} \circ \varphi_1$ is defined.

Sormani-Wenger: Intrinsic Flat Distance

between integral current spaces $M_i^m = (X_i, d_i, T_i)$ is:

$$d_{SWIF}(M_1^m, M_2^m) = \inf\left\{ d_F^Z\left(\varphi_{1\#}T_1, \varphi_{2\#}T_2\right) \mid \varphi_i : M_i^m \to Z \right\}$$

where $\varphi_\# T(f, \pi_1, ..., \pi_m) = T(f\circ\varphi, \pi_1\circ\varphi, ..., \pi_m\circ\varphi)$, and where the infimum is taken over all complete metric spaces, Z, and over all distance preserving maps $\varphi_i : M_i^m \to Z$.

Here: $d_F^Z\left(\varphi_{1\#}T_1, \varphi_{2\#}T_2\right)$ is the Federer-Fleming Flat dist

$$= \inf\left\{ \underset{area}{\mathbf{M}}(A) + \underset{vol}{\mathbf{M}}(B) : A + \partial B = \varphi_{1\#}T_1 - \varphi_{2\#}T_2 \right\}$$

Thm [SW-JDG]: If M_i are compact and $d_{SWIF}(M_1, M_2) = 0$ then $\exists$ a current preserving isometry $F : M_1 \to M_2$: $d_2(F(p), F(q)) = d_1(p,q)\ \forall p, q \in X_1$ and $F_\# T_1 = T_2$.

Pf: Show inf achieved: $\exists \varphi_i : M_i \to Z$ s.t. $\varphi_{1\#}T_1 = \varphi_{2\#}T_2$.
So $\mathrm{set}(\varphi_{1\#}T_1) = \mathrm{set}(\varphi_{2\#}T_2)$ and $F = \varphi_2^{-1}\circ\varphi_1$ is defined.

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Take $B = [[\mathbb{S}_+^{m+1}]]$ so $\partial B = \varphi_{1\#}[[\mathbb{S}^m]]$ and $A = 0$. $\square$

Implications of SWIF Convergence [SW-JDG]

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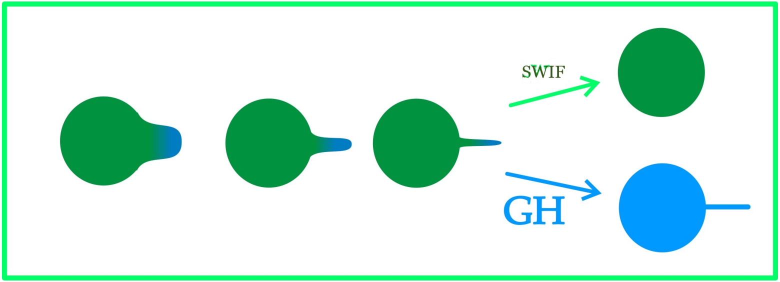
$$M_j \xrightarrow{\text{SWIF}} M_\infty \implies \liminf_{j\to\infty} \mathbf{M}(M_j) \ge \mathbf{M}(M_\infty)$$

SWIF Compactness Theorems

Thm [SW]: If $M_j \xrightarrow{\text{GH}} M_{GH}$ and $\mathrm{Vol}(M_j) \le V_0$ and $\mathrm{Vol}(\partial M_j) \le A_0$ then $\exists M_{j_k} \xrightarrow{\text{SWIF}} M_{SWIF}$ where $M_{SWIF} \subset M_{GH}$ or $M_{SWIF} = 0$.



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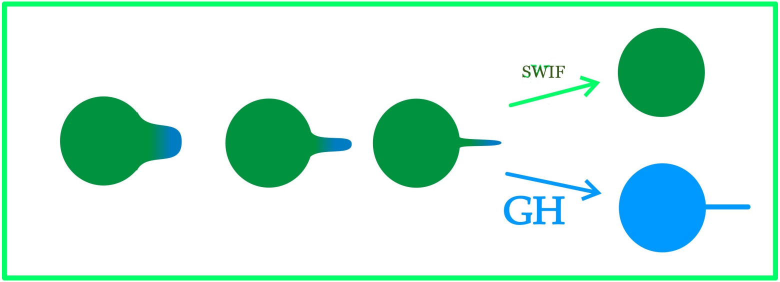

Proof: By Gromov's Compactness Thm, $\exists$ compact Z and dist pres maps $\varphi_j : M_j \to Z$ s.t. $d_H^Z(\varphi_j(M_j), \varphi_{GH}(M_{GH})) \to 0$.

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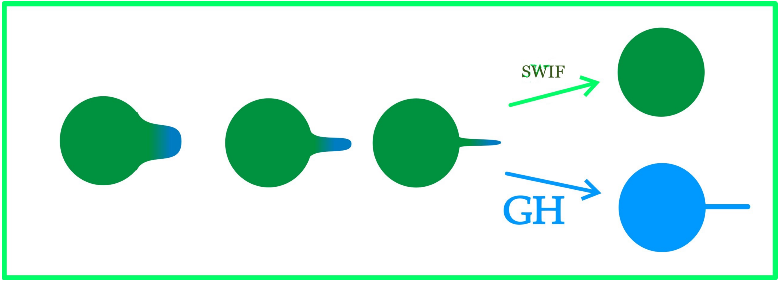

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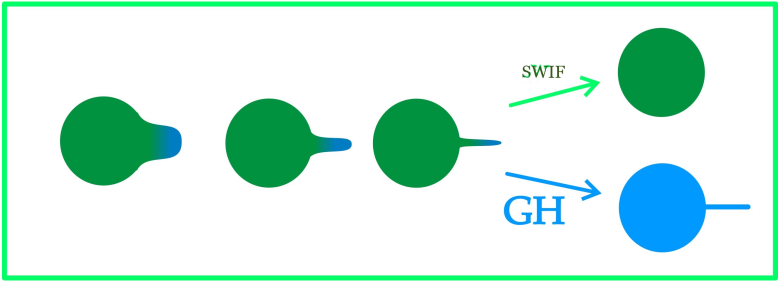

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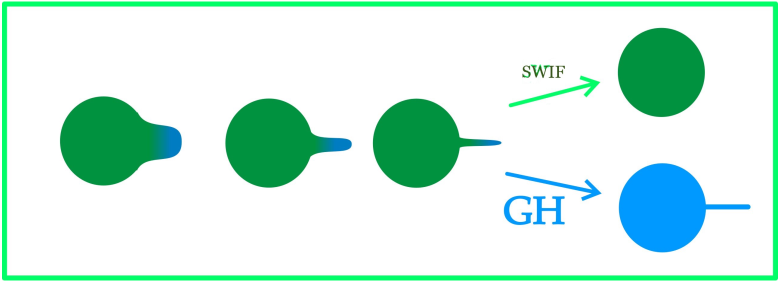

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$\text{set}(T_\infty) \subset \varphi_{GH}(M_{GH}) \subset Z$. Let $M_{SWIF} = (\text{set}(T_\infty), d_Z, T_\infty)$ $\square$.

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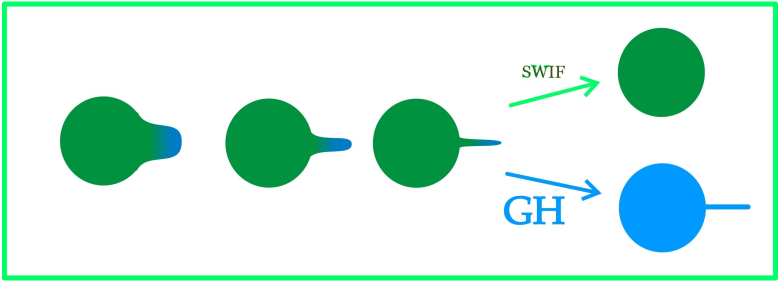

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Wenger Compactness Thm: If $\text{Diam}(M_j) \le D$ and $\mathbf{M}(M_j) \le V$ and $\mathbf{M}(\partial M_j) \le A_0$ then $\exists M_{j_k} \xrightarrow{\text{SWIF}} M_{SWIF}$ possibly $M_{SWIF} = 0$.

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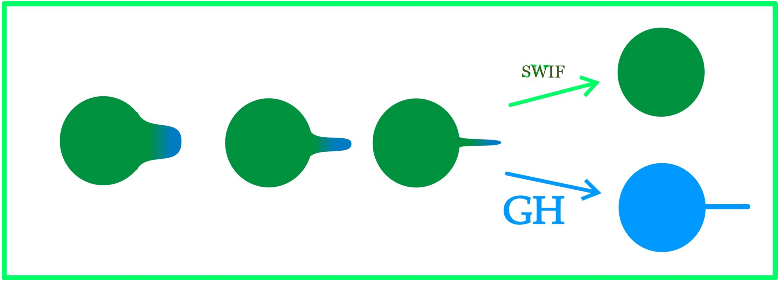

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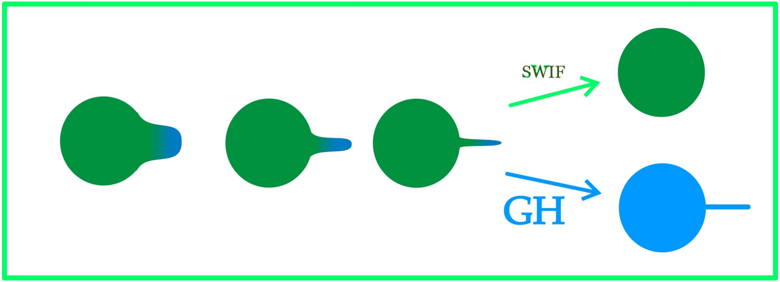
Wenger Compactness Thm: If $\mathrm{Diam}(M_j) \le D$ and $\mathbf{M}(M_j) \le V$ and $\mathbf{M}(\partial M_j) \le A_0$ then $\exists M_{j_k} \xrightarrow{\text{SWIF}} M_{SWIF}$ possibly $M_{SWIF} = 0$.

SWIF Compactness Theorems

Thm [SW]: If $M_j \xrightarrow{\text{GH}} M_{GH}$ and $\text{Vol}(M_j) \le V_0$ and $\text{Vol}(\partial M_j) \le A_0$ then $\exists M_{j_k} \xrightarrow{\text{SWIF}} M_{SWIF}$ where $M_{SWIF} \subset M_{GH}$ or $M_{SWIF} = 0$.



Proof: By Gromov's Compactness Thm, $\exists$ compact Z and dist pres maps $\varphi_j : M_j \to Z$ s.t. $d_H^Z(\varphi_j(M_j), \varphi_{GH}(M_{GH})) \to 0$.
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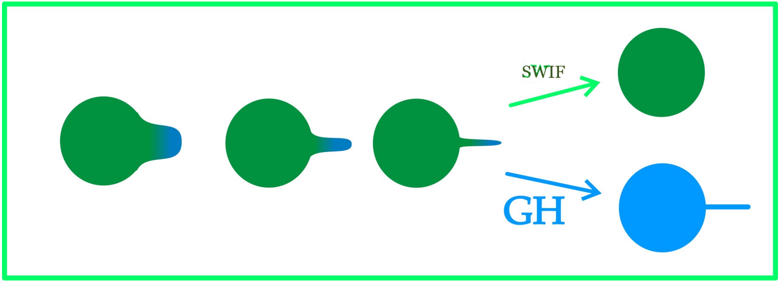

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How do we know which regions disappear? Use Filling Volumes!

Adapting Gromov's Filling Volume [Portegies-Sormani]:

$$\mathbf{FillVol}(M^m) = \inf\left\{\mathbf{M}(N^{n+1}) \,|\, \partial N^{n+1} = M^m\right\}$$

where the inf is over integral current spaces $N^{n+1} = (X_N, d_N, T_N)$ such that $\exists$ current preserving isometry $F : M^m \to \partial N^{n+1}$.

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Pf: Perelman Colding Gv: $\exists C^m_{H,V}$ s.t. $\mathbf{FillVol}(\partial B(p,r)) \ge C^m_{V,H} r^m$. combined with Corollary above and Portegies-Sormani (next slide) which says $B_j \xrightarrow{\mathrm{SWIF}} B_\infty \implies \mathbf{FillVol}(\partial B_j) \to \mathbf{FillVol}(\partial B_\infty)$. $\square$

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Thm: If $B(p, r)$ is a ball in an integral current space M then for a.e. $r > 0$ $(B(p, r), d_M, [[B(p, r)]])$ is an integral current space and so is $\partial(B(p, r), d_M, [[B(p, r)]]) = (\partial B(p, r), d_M, [[\partial B(p, r)]])$, and

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Recall: $p \in \mathrm{set}(T)$ if $\liminf_{r\to 0} \mathbf{M}(B(p, r))/r^m > 0$.
Coro: $p \in \mathrm{set}(T)$ if $\liminf_{r\to 0} \mathbf{FillVol}(\partial B(p, r))/r^m > 0$.
This corollary was applied by S-Wenger Matveev-Portegies to prove
Thm: $M_{GH} = M_{SWIF}$ for M_j with $\mathrm{Vol}(M_j) \ge V$ and $\mathrm{Ricci} \ge H$.
Pf: Perelman Colding Gv: $\exists C^m_{H,V}$ s.t. $\mathbf{FillVol}(\partial B(p, r)) \ge C^m_{V,H} r^m$.
combined with Corollary above and Portegies-Sormani (next slide)
which says $B_j \xrightarrow{\text{SWIF}} B_\infty \implies \mathbf{FillVol}(\partial B_j) \to \mathbf{FillVol}(\partial B_\infty)$. $\square$

Filling Volumes and Balls [Portegies-Sormani]:

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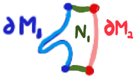

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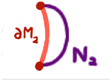

4.

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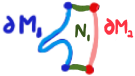

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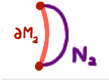

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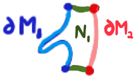

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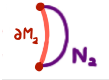

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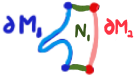

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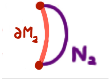

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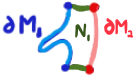
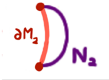
2. 

3. 

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Intrinsic Flat and Gromov-Hausdorff Convergence

Christina Sormani

CUNY GC and Lehman College

Lectures IV: Proving Intrinsic Flat Convergence

Volume Preserving Intrinsic Flat $\mathcal{VF}$ Convergence

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Defn: $M_j \xrightarrow{\mathcal{VF}} M_\infty$ if $M_j \xrightarrow{\mathcal{F}} M_\infty$ and $\mathrm{Vol}(M_j) \to \mathrm{Vol}(M_\infty)$.

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Defn: $M_j \xrightarrow{\mathcal{F}} M_\infty$ if $d_{\mathcal{F}}(M_j, M_\infty) = d_{SWIF}(M_j, M_\infty) \to 0$:

Sormani-Wenger: Intrinsic Flat Distance

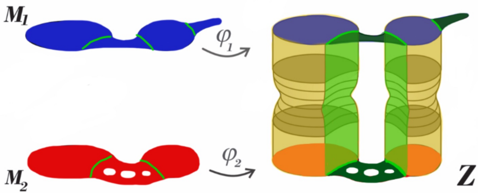
The *intrinsic flat distance* between oriented manifolds M_i^m is:

$$d_{SWIF}(M_1^m, M_2^m) = \inf\left\{ d_F^Z\left(\varphi_{1\#}[[M_1^m]], \varphi_{2\#}[[M_2^m]]\right) \mid \varphi_i : M_i^m \to Z \right\}$$

where the infimum is taken over all complete metric spaces, Z,
and over all distance preserving maps $\varphi_i : M_i^m \to Z$.

Here: $d_F^Z\left(\varphi_{1\#}[[M_1^m]], \varphi_{2\#}[[M_2^m]]\right)$ is the Federer-Fleming Flat dist

$$= \inf\left\{ \underset{area}{\mathbf{M}}(A) + \underset{vol}{\mathbf{M}}(B) : A + \partial B = \varphi_{1\#}[[M_1^m]] - \varphi_{2\#}[[M_2^m]] \right\}$$



Lakzian-Sormani: Estimating d_{SWIF}

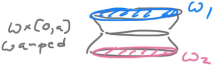
Lakzian-Sormani: Suppose (M_1, g_1) and (M_2, g_2) are oriented precompact Riemannian manifolds with diffeomorphic subregions $W_i \subset M_i$. Identifying $W_1 = W_2 = W$ assume that on W we have

$$g_1 \le (1+\varepsilon)^2 g_2 \text{ and } g_2 \le (1+\varepsilon)^2 g_1.$$

Taking the extrinsic diameters,

$$\operatorname{diam}(M_i) \le D$$

we define a hemispherical width,



$$a > \frac{\arccos(1+\varepsilon)^{-1}}{\pi} D.$$

to find dist pres maps

g on W×[0,a] is $g = dr^2 + f^2(r) g_i$

$f(r) = \max\{\cos(r), \cos(a-r)\}$

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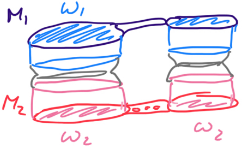

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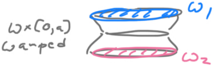

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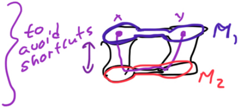
Taking the difference in distances with respect to the outside manifolds, we set

$$\lambda = \sup_{x,y\in W} |d_{M_1}(x,y) - d_{M_2}(x,y)| \leq 2D,$$



and we define the height,

$$\bar{h} = \max\{\sqrt{2\lambda D}, D\sqrt{\varepsilon^2 + 2\varepsilon}\}.$$

Then taking $Z = M_1 \sqcup W_1 \times [0,h] \sqcup W\times[0,a]_{\text{warped}} \sqcup W_2 \times [0,h] \sqcup M_2$

$$d_{\mathcal{F}}(M_1, M_2) \leq (2\bar{h} + a)\Big(\operatorname{Vol}_m(W_1) + \operatorname{Vol}_m(W_2) + \operatorname{Vol}_{m-1}(\partial W_1) + \operatorname{Vol}_{m-1}(\partial W_2)\Big)$$
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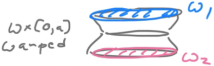

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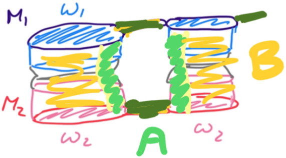

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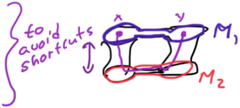

and we define the height,

$$\bar{h} = \max\{\sqrt{2\lambda D}, D\sqrt{\varepsilon^2 + 2\varepsilon}\}.$$

Then taking $Z = M_1 \sqcup W_1 \times [0,h] \sqcup \underset{\text{warped}}{W \times [0,a]} \sqcup W_2 \times [0,h] \sqcup M_2$

$$d_{\mathcal{F}}(M_1, M_2) \le (2\bar{h} + a)\Big(\mathrm{Vol}_m(W_1) + \mathrm{Vol}_m(W_2) + \mathrm{Vol}_{m-1}(\partial W_1) + \mathrm{Vol}_{m-1}(\partial W_2)\Big)$$
$$+ \mathrm{Vol}_m(M_1 \setminus W_1) + \mathrm{Vol}_m(M_2 \setminus W_2),$$

Allen-Perales-Sormani VADB

Allen-Perales-Sormani: [arXiv:2003.01172]

$$M_j \xrightarrow{\text{VADB}} M_\infty \implies M_j \xrightarrow{\mathcal{VF}} M_\infty.$$

Defn: *Volume Above Distance Below Conv:* $M_j \xrightarrow{\text{VADB}} M_\infty$ if $Vol_j(M_j) \to Vol_\infty(M_\infty)$ and $\exists D > 0$ s.t. $Diam(M_j) \le D$ and $\exists$ C^1 *diffeomorphism* $\psi_j : M_\infty \to M_j$ such that

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An earlier theorem that inspired us:

Huang-Lee-Sormani: *Given (M, d_0) Riemannian without boundary and fix $\lambda > 0$, suppose that d_j are length metrics on M such that*

$$\lambda \geq \frac{d_j(p,q)}{d_0(p,q)} \geq \frac{1}{\lambda}.$$

Then there exists a subsequence, also denoted d_j, and a length metric d_∞ such that d_j converges uniformly to d_∞:

$$\varepsilon_j = \sup\{|d_j(p,q) - d_\infty(p,q)| : \ p, q \in X\} \to 0.$$

and M_j converges in the intrinsic flat and Gromov-Hausdorff sense to M_∞:

$$M_j \xrightarrow{\mathcal{F}} M_\infty \text{ and } M_j \xrightarrow{GH} M_\infty$$

where $M_j = (M, d_j)$ and $M_\infty = (M, d_\infty)$.

Allen-Perales-Sormani VADB Constructing Z

Allen-Perales-Sormani *Let M be an oriented, connected and closed manifold, $M_j = (M, g_j)$ and $M_0 = (M, g_0)$ be Riemannian manifolds with $\mathrm{Diam}(M_j) \le D$, $\mathrm{Vol}_j(M_j) \le V$ and $F_j : M_j \to M_0$ a C^1 diffeomorphism and distance non-increasing map:*

(120) $$d_j(x,y) \ge d_0(F_j(x), F_j(y)) \quad \forall x, y \in M_j.$$

Let $W_j \subset M_j$ be a measurable set and assume that there exists a $\delta_j > 0$ so that

(121) $$d_j(x,y) \le d_0(F_j(x), F_j(y)) + 2\delta_j \qquad \forall x, y \in W_j$$

with

(122) $$\mathrm{Vol}_j(M_j \setminus W_j) \le V_j$$

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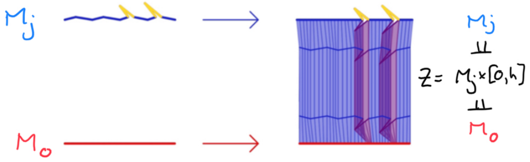

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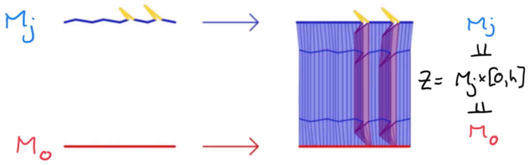

Z is M_j glued along W_j to $M_j \times [0, h]$ glued along $F_j(W_j)$ to M_0.

Allen-Sormani VADB to ptwise a.e. on $M \times M$

Allen-Sormani: *If (M, g_j) are compact continuous Riemannian manifolds without boundary and (M, g_0) is a smooth Riemannian manifold such that*

(85) $$g_j(v,v) \geq g_0(v,v) \qquad \forall v \in T_pM$$

and

(86) $$\mathrm{Vol}_j(M) \to \mathrm{Vol}_0(M)$$

then there exists a subsequence such that

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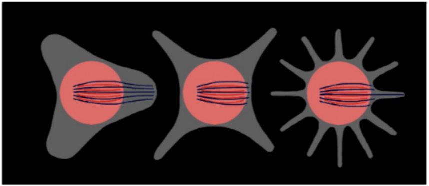

FIGURE 2. A tube $\mathcal{T}$ foliated by g_0-geodesics, γ, with $L_j(\gamma) \geq L_0(\gamma)$ has $\mathrm{Vol}_j(\mathcal{T}) \to \mathrm{Vol}_0(\mathcal{T})$ so $L_j(\gamma) \to L_0(\gamma)$ for almost every γ but not for γ ending at a tip.

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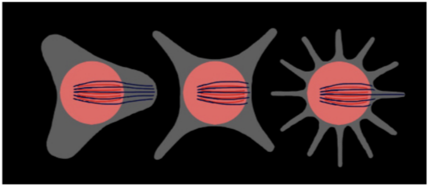

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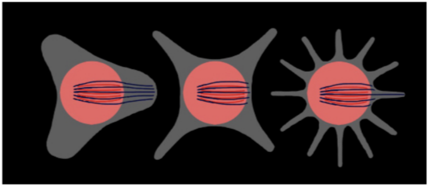

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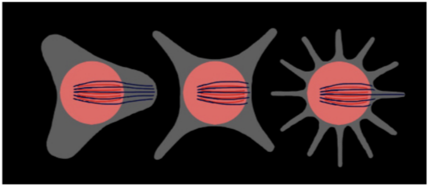

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Egoroff's Theorem? But Egoroff's Theorem only gives a set $S \in M \times M$ controlling $d(p,q)$ uniformly $\forall (p,q) \in S$...

Allen-Perales-Sormani Ptwise to Uniform on $W \subset M$

Now we apply Egoroff's theorem to obtain uniform convergence on a set of almost full measure.

Proposition . *Under the hypotheses of Theorem* [4.1], *for every* $\varepsilon > 0$ *there exists a* $dvol_{g_0} \times dvol_{g_0}$ *measurable set,* $S_\varepsilon \subset M \times M$*, such that*

$$\sup\{|d_j(p,q) - d_0(p,q)| \,:\, (p,q) \in S_\varepsilon\} = \delta_{\varepsilon,j} \to 0, \tag{185}$$

$$\mathrm{Vol}_{0\times 0}(S_\varepsilon) > (1-\varepsilon)\,\mathrm{Vol}_{0\times 0}(M \times M). \tag{186}$$

and

$$(p,q) \in S_\varepsilon \iff (q,p) \in S_\varepsilon. \tag{187}$$

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and

$$S_{p,\varepsilon} = \{q \in M : (p,q) \in S_\varepsilon\},$$

are $dvol_{g_0}$ measurable and satisfy

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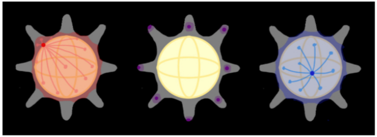

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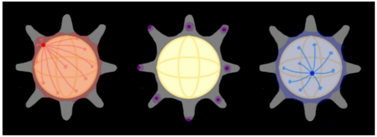

Lemma *For* $W_{\kappa\varepsilon} = \{p : \mathrm{Vol}_0(S_{p,\varepsilon}) > (1-K\varepsilon)\mathrm{Vol}_0(M)\}$

$$\mathrm{Vol}_0(W_{\kappa\varepsilon}) > \frac{\kappa - 1}{\kappa}\,\mathrm{Vol}_0(M).$$

and $|d_j(p,q) - d_0(p,q)| < \delta_{\varepsilon,j} \quad \forall p,q \in W_{K,\varepsilon}$

Allen-Perales-Sormani VADB to $\mathcal{VF}$ is Proven

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combined with our estimate on SWIF:

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completes the proof of $M_j \xrightarrow{\text{VADB}} M_\infty \implies M_j \xrightarrow{\mathcal{VF}} M_\infty$. $\square$

SWIF limits are Integral Current Spaces

Recall Flat limits of oriented submanifolds are integral currents:

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Ambrosio-Kirchheim (2000): an integral current, T, on Z is an integer rectifiable current s.t. ∂T is also integer rectifiable.

Defn: an integer rectifiable current, T, has cntbly many pairwise disjoint biLip charts $\varphi_i : A_i \to \varphi_i(A_i) \subset Z$ and weights $a_i \in \mathbb{Z}$ s.t.

$$T(f, \pi_1, ..., \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} (f \circ \varphi)\, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

with mass $\mathbf{M}(T) = ||T||(Z)$ where $||T|| = \lambda\, \theta\, (\mathcal{H}_m \llcorner \mathrm{set}\, T)$ and

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Furthermore: $\mathbf{M}(M) = \mathbf{M}(T)$ and $\partial M = (\mathrm{set}(\partial T), d, \partial T)$.
Thus X is cntbly $\mathcal{H}^m$ rectifiable: it has cntbly many pairwise disjoint Lip charts $\varphi_i : A_i \to X$ s.t. $\mathcal{H}^m(X \setminus \bigcup_{i=1}^{\infty} \varphi_i(A_i)) = 0$.

Oriented Riemannian Mnflds and Integral Current Spaces

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Sormani-Wenger: Intrinsic Flat Distance

between integral current spaces $M_i^m = (X_i, d_i, T_i)$ is:

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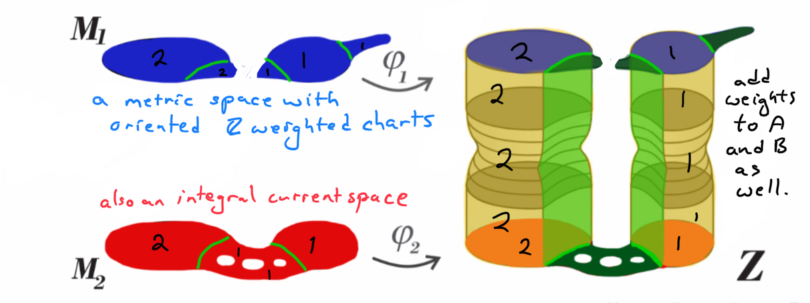
where the infimum is taken over all complete metric spaces, Z,
and over all distance preserving maps $\varphi_i : M_i^m \to Z$.

Here: $d_F^Z\left(\varphi_{1\#} T_1, \varphi_{2\#} T_2\right)$ is the Federer-Fleming Flat dist

$$= \inf \left\{ \underset{area}{\mathbf{M}}(A) + \underset{vol}{\mathbf{M}}(B) : A + \partial B = \varphi_{1\#} T_1 - \varphi_{2\#} T_2 \right\}$$



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$$M_j \xrightarrow{\text{SWIF}} M_\infty \implies \liminf_{j\to\infty} \mathbf{M}(M_j) \geq \mathbf{M}(M_\infty)$$

Thm [Sor-ArzAsc]: For any $p \in M_\infty$ there exists $p_j \in M_j$ s.t.
$d_Z(\varphi_j(p_j), \varphi_\infty(p)) \to 0$.

Arzela-Ascoli Theorem

Theorem [Sor-ArzAsc] *Suppose $M_i = (X_i, d_i, T_i)$ are integral current spaces for $i \in \{1, 2, ..., \infty\}$ and $M_i \xrightarrow{\mathcal{F}} M_\infty$ and $F_i : X_i \to W$ are Lipschitz maps into a compact metric space W with*

(188) $$\mathrm{Lip}(F_i) \le K,$$

then a subsequence converges to a Lipschitz map $F_\infty : X_\infty \to W$ with

(189) $$\mathrm{Lip}(F_\infty) \le K.$$

More specifically, there exists isometric embeddings of the subsequence, $\varphi_i : X_i \to Z$, such that $d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty) \to 0$ and for any sequence $p_i \in X_i$ converging to $p \in X_\infty$,

(190) $$d_Z(\varphi_i(p_i), \varphi_\infty(p)) \to 0,$$

one has converging images,

(191) $$d_W(F_i(p_i), F_\infty(p)) \to 0.$$

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Thm: If $B(p, r)$ is a ball in an integral current space M then $(B(p, r), d_M, [[B(p, r)]])$ is an integral current space for a.e. $r > 0$.

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Thm: If $B(p,r)$ is a ball in an integral current space M then $(B(p,r), d_M, [[B(p,r)]])$ is an integral current space for a.e. $r > 0$.

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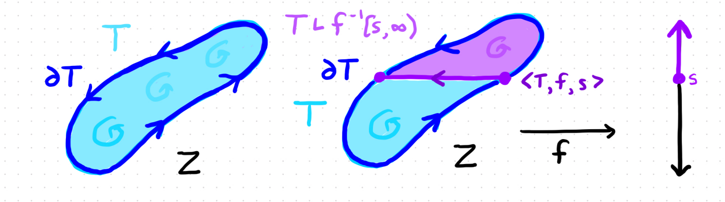
The proofs of the above use the Ambrosio-Kirchheim Slicing Thm.

Ambrosio-Kirchheim Slicing Theorem:

Given Lipschitz, $f : Z \to \mathbb{R}$, and integral current, T, for a.e. $s \in \mathbb{R}$ one can define the slice of T by f at s which is an integral current

$$< T, f, s > := -\partial \left(T \llcorner f^{-1}(s, \infty) \right) + (\partial T) \llcorner f^{-1}(s, \infty),$$

where S restricted to U is $(S \llcorner U)(h, \pi_1, ...) = S(\chi_U \cdot h, \pi_1, ...)$.



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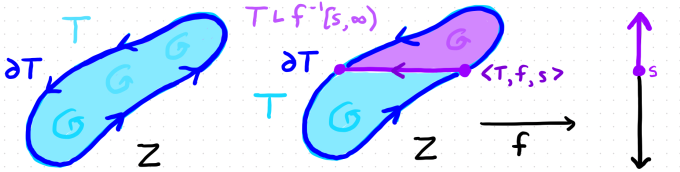

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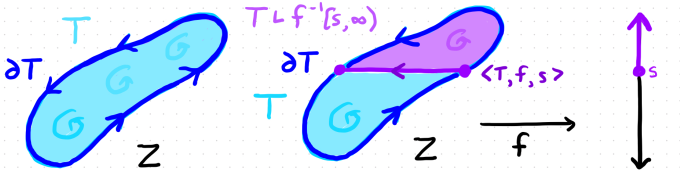

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$$\int_{s\in\mathbb{R}} \mathbf{M}(< T, f, s >)\, ds = \mathbf{M}(T \llcorner df) \leq Lip(f)\, \mathbf{M}(T)$$

where $(T \llcorner df)(h, \pi_1, ..., \pi_{m-1}) = T(h, f, \pi_1, ...\pi_{m-1})$.

Flat Distance between Slices in Z

Given integral currents T_i in Z
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Since

$$\int_{s \in \mathbb{R}} \mathbf{M}(< A, f, s >) \, ds \leq Lip(f) \, \mathbf{M}(A)$$

and

$$\int_{s \in \mathbb{R}} \mathbf{M}(< B, f, s >) \, ds \leq Lip(f) \, \mathbf{M}(B)$$

we have,

$$\int_{s \in \mathbb{R}} d_F^Z(< T_1, f, s >, < T_2, f, s >) \, ds \leq Lip(f)(\mathbf{M}(A) + \mathbf{M}(B))$$

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If $d_F^Z(T_j, T_\infty) \to 0$ and $f : Z \to \mathbb{R}$ has $Lip(f) \le 1$ then

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[PS] also define a sliced filling volume and estimate it.

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This implies $\mathbf{M}(M_\infty) \geq \liminf_{j\to\infty} \mathbf{M}(M_j)$

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IAS Emerging Topic Conjecture: Tan Cones

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So a subsequence $M_j \xrightarrow{\text{SWIF}} M_\infty$.

Conjecture: If in addition we have $Scalar_j \ge 0$ and $MinA_j \ge A$ where $MinA_j = min\{Area(\Sigma) : \text{closed min surfaces } \Sigma \subset M_j^3\}$

Then M_∞ has *generalized* "$Scalar \ge 0$"

Furthermore: we believe that we have $M_j \xrightarrow{\mathcal{VF}} M_\infty$ where M_∞ is a connected length space with Euclidean tangent cones.

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By Ambrosio-Kirchheim theory, we know that at a.e. $p \in M_\infty$, there is a tangent cone, T_pM, which is a normed vector space:

$$(B(p, r_i), d/r_i, [[B(p, r_i)]]) \xrightarrow{\text{SWIF}} B(0, 1) \subset T_pM.$$

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Open: Prove $||T_\infty|| = \mathbb{H}^3$. (Ricci case by Colding "Volumes....).