Introduction to Brakke flows I

Today:
1. Smooth mean curvature flows: basic properties
2. Weak compactness for submanifolds
3. Brakke flows: definition & basic properties

Mean curvature flow

$M^n$ closed, $\overrightarrow{T}: M^\times [0,T) \rightarrow \mathbb{R}^{n+1}$ smooth family of immersions, $M_t = T(M,t)$, $(M_t)_{0 \leq t \leq T}$ solves mean curvature flow

\[
\frac{d}{dt} \frac{\vec{X}}{\dot{H}} = -H \cdot \nu \quad H = \frac{dM}{dt} = \frac{\partial M}{\partial t}
\]

$T(\cdot, 0) = T_0$

Examples:

- shrinking spheres $S^n_{R(t)}$, $R(t) = \sqrt{R^2 - 2nt}$

- $-n$-cylinder $S^n_{R(t)} \times \mathbb{R}^{n-k}$, $R(t) = \sqrt{R^2 - 2kt}$

- gnomes

\[ y(t) = -\ln \cos (\tau) \]

- eternal translating

- paper chip

Thus (Gage-Hamilton '86, Grayson '87): Curve shortening flows contract simple, closed curves in $\mathbb{R}^2$ to "round points" in finite time.
The (Huisken '84) HCT contracts closed, convex hypersurfaces in $\mathbb{R}^{n+1}$ in finite time to 'round points'.

**Basic properties:**

- For a flow with normal speed $-t\cdot N$, we have
  \[
  \frac{d}{dt} |M| = \frac{d}{dt} \int 1 dm = -\int t \cdot H dm \\
  \Rightarrow \frac{d}{dt} |M| = -\int t \cdot H dm \text{ in HCT (L^- quadrilateral flow)}
  \]

- Smooth short-time existence

- Avoidance principle: assume $(M^+_t)_{t \in \mathbb{R}_+}, (M^+_t)_{t \in \mathbb{R}_+}$, two HCTs s.t. $M^+_0 \cap M^-_0 = \emptyset$
  \[
  \Rightarrow M^+_0 \cap M^-_0 = \emptyset \quad \forall \, t \in \mathbb{R}_+
  \]

- Embeddedness preserved

- Finite existence time

- Singularities
Weak compactness for submanifolds

Recall the natural estimate

\[ \int_0^1 \int_{\mathbb{R}} |H|^2 \, dt \, dx \leq MH \]

Q: compactness of submanifolds under weak mean curvature bounds?

Let \( (M_i) \) sequence of \( m \)-submanifolds of \( \mathbb{R}^n \),

\[ \mathcal{F}^m(M_i \cap K) \leq C(K) \quad \forall K \subset \mathbb{R}^n. \]

Assume

\[ \int (H^2) \, d\mu_i \leq C(K) \quad \forall i \]

Can we understand in what sense

\[ M_i \rightarrow M_0 \quad ? \]

What structure does \( M_0 \) have?

Vanishing:

\( M \subset \mathbb{R}^n \) \( m \)-submanifold, get a Radon measure

\[ \nu_M \quad \mu_M(s) = \mathcal{F}^m(M \cap s) \]

\[ \int d\mu_M := \int \alpha d\mathcal{F}^m \quad \alpha \in C_c^\infty (\mathbb{R}^n) \]

So if \( (M_i) \) as above we have (up to a subsequence)

\[ \mu_{M_i} \rightarrow \mu \]

But: quite coarse, so wish to refine it.

Note: It defines a Radon measure on \( \mathbb{R}^n \times \mathcal{G}(m,n) \),
the Grassmann bundle of \( m \)-planes in \( \mathbb{R}^n \).
Define \( V_m \)
\[
\int f \, dV_m = \int f(x, \text{Tan}(m, x)) \, d\mathcal{H}^m
\]
for \( f \in C_0^\infty(\mathbb{R}^n \times G(m, n)) \), not
\[
V_m(s) = \mathcal{H}^m(\{x \in m : (x, \text{Tan}(m, x)) \in S\})
\]
Can assume \( V_{m_i} \to V \). Note \( \pi^*_m V_m = M_m \)

Def: An \( m \)-dim. manifold \( m \) is a Baire measure on \( \mathbb{R}^n \times G(m, n) \).

Examples of manifold convergence: work in \( \mathbb{R}^2 \)

i) \( M_n \to m \cdot \frac{1}{2} \mu_{c_{01,1}} \)
\[
V_{M_n} \to \frac{1}{2} V_{c_{01,1}}
\]

ii) \( \frac{1}{n} \) intervals
\[
M_n \to \mu_{c_{01,1}} / V_{M_n} \to V_{c_{01,1}}
\]

iii) \( M_n \)
\[
M_n \to \frac{1}{2} \mu_{\text{dsg}} \quad \text{but } \exists k > 0 \text{ s.t.} \quad V_{M_n} \to k V_{\text{dsg}}
\]

**Integral Manifolds**

Lemma: Let \( M, M' \) be \( m \)-dim. \( C^\infty \) submanifolds of \( \mathbb{R}^n \).

Let \( \mathcal{Z} = \{x \in M \cap M' : \text{Tan}(M, x) = \text{Tan}(M', x)\} \).

Then \( \mathcal{Z}^\mu(\mathcal{Z}) = 0 \).
\textbf{To}: If $M, M'$ are hypersurfaces, the $T$ is a $(m-1)$-dim $C^1$ submanifold by transversality. In higher codimension, one may show that $S \cap M, M'$ is a manifold in a $(m-1)$-dim $C^1$ submanifold, by projection to a linear subspace (reasoning).

\textbf{Cor:} Assume $S \subseteq U_{M_i}$ and $S \subseteq U_{M_j}$ for $M_i, M_j$, $m$-dim submanifolds of $\mathbb{R}^n$. Define $T$ on $S$ by

$$T(x) = \text{Tan}(M_i, x)$$

where $i$ is the first $i \in M_i \setminus U_{M_j}$, $x \in M_i \setminus U_{M_j}$, $j \in i$.

Define $T'$ similarly. Then

$$T(x) = T'(x)$$

for all $x \in S$.

Thus for $S$, a Borel subset of a $C^1$ submanifold of $\mathbb{R}^n$, we define $V$, to be the manifold given by

$$\int f dV = \int f(x, \text{Tan}(M_i, x)) \, d\text{Vol}(x)$$

This does not depend on the choice of $M_i$ by the previous discussion.

\textbf{Def:} An \textit{integral submanifold} is a submanifold which can be written as

$$V = \bigcup_{i=1}^{n} V_i.$$
\[ f(x) = \int_{V} v(x, \nabla \phi(x)) \, dv \]
**Rem:** \( X \in C_c^1(\mathbb{R}^N; \mathbb{R}^N) \), \( \phi_\varepsilon \) - s.c. \( \text{ s.t. } \phi_0 = 1 \text{d}, \)
\( \frac{d}{dt} \phi_\varepsilon = X \cdot \phi_\varepsilon \), hence
\[
\frac{d}{dt} \left( (\phi_\varepsilon)_\# V(x) \right) \bigg|_{t=0} = \delta V(x)
\]
\( \text{supp } X \subset K \subset \subset \mathbb{R}^N \)

**Note:** If \( V_\varepsilon \to V \), then \( \delta V_\varepsilon(x) \to \delta V(x) \).

Assume we have local bounds on the first variation, i.e.,
\[
\| \delta V(x) \| \leq C_K \| x \|_0
\]

If \( \text{supp } X \subset K \subset \subset \mathbb{R}^N \)

**Real repr. for:** \( \int \text{Radon measure } \lambda \text{ ad } A \)-measurable

\[
\delta V(x) = \int \langle x, \lambda \rangle \ dA
\]

Decompose \( A \) w.r.t. \( \mu_N \), then it \( \lambda_{ac} \ll \mu_N \text{ and } \lambda_{ij} \) s.t.
\[
\delta V(x) = \int \langle x, \lambda \rangle \ d\lambda_{ac} + \int \langle x, \lambda \rangle \ d\lambda_{ij}
\]
\[
= \int \langle x, \lambda \rangle d\lambda_{ac} + \int \langle x, \lambda \rangle \ d\lambda_{ij} = \int \langle x, \lambda \rangle d\lambda_{ac} + \int \langle x, \lambda \rangle \ d\lambda_{ij}
\]
\[
\Rightarrow \delta V(x) = - \int \langle x, \lambda \rangle \ d\mu_N + \int \langle x, \eta \rangle \ d\lambda_{ij}
\]
The (Allard's compactness theorem)

Suppose \( V_i \to V \) is a sequence of integral varifolds converging weakly to a varifold \( V \). If the \( V_i \) have locally uniformly bounded first variation, i.e.,
for \( k \in \mathbb{N} \) there is \( C_k \) (depend. of \( k \)) s.t.
\[ \forall x \in C^1_c(\mathbb{R}^n; \mathbb{R}^n) \quad \text{with} \quad \text{supp}(x) \subset \text{ker} \quad \text{we \ have} \]
\[ |\delta V_i(x)| \leq C_k \|x\|_{\infty} \]

then \( V \) is also an integral varifold.

Rem.: 
1) We trivially obtain \( |\delta V(x)| \leq C_k \|x\|_\infty \)

2) A sequence of smooth \((C^1)\)-submanifolds satisfies the hypotheses of Allard's theorem iff
\[ \int |H_i| \, d\mathcal{H}_n + \sum_k \int |\partial \Omega_i| \leq C_k \]
where \( \partial \Omega_i \) is the boundary measure of \( \Omega_i \).

\[ \int \quad 2 \quad \ast \]

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1) The quantities '\|H_i\|' and '\|\partial \Omega_i\|' can be mixed up in the limit.

2) \[
\begin{array}{ccc}
\text{no}
\end{array}
\]
**Bakelis flows**

Def: An (n-dim) inital Bakelis flow in \( \mathbb{R}^{n+1} \) is a 1-parameter family of Borel measures \( \{\mu(t)\} \subseteq \mathcal{M}(\mathbb{R}^n) \) for some interval \( I \subseteq \mathbb{R} \).

(i) For almost all \( t \in I \), there exists an immersed \( n \)-manifold \( V(t) \) with \( \mu(t) = \mu(V(t)) \) st. \( V(t) \) has

locally bounded first variation and has weak mean curvature \( \frac{H}{2} \) orthogonal to \( T(V(t), \cdot) \) almost every.

(ii) For a bounded interval \( [a,b] \subseteq I \) and any compact \( \Theta \subseteq \mathbb{R}^n \),

\[
\int_{a}^{b} \int_{\Theta} (1 + |\nabla|^2) \, d\mu(t) \, dt < \infty
\]

(3) If \( \{[a(t),b(t)] \subseteq I \) ad \( \mathbf{f} \in C^2(\mathbb{R}^{n+1} \times \mathbb{R}, \mathbb{R}^n) \) for \( t > 0 \), then

\[
\int_{a}^{b} \int_{\Theta} \mathbf{f}(\cdot,t) \, d\mu(t) \, dt - \int_{a}^{b} \int_{\Theta} \mathbf{f}(\cdot,b(t)) \, d\mu(t) \leq \int_{a}^{b} \left( -|H|^2 + \frac{\langle \mathbf{H}, \nabla \mathbf{f} \rangle}{5} \right) \, d\mu(t) \, dt
\]
Ren: If \((H_t)_{t \geq 0}\) is a smooth \(H_t\)

\[
\frac{d}{dt} \int_{H_t} dA = \int_{H_t} \left( -1 \frac{\partial t}{\partial t} + \langle \nabla t, \nabla t \rangle + \frac{\partial t}{\partial t} \right) dA
\]

So why do we only require the inequality \(\circ (3)\)?

En amn \(f \in W^{1,2}(\Omega)\), \(\int |\nabla f|^2 dx \leq C\)

Ca amn \(f \rightarrow f_0\)

\[
\int |\nabla f_m|^2 dx \leq \liminf_{m \rightarrow \infty} \int |\nabla f_m|^2
\]