# Introduction to Brakke flows I

**Today:**

1. Smooth mean curvature flows: basic properties
2. Weak compactness for submanifolds
3. Brakke flows: definition & basic properties

## Mean curvature flow

$M^n$ closed, $F: M^n \times [0,T) \to \mathbb{R}^{n+1}$ smooth family of immersions, $M_t = F(M,t)$. $(M_t)_{0 \le t < T}$ solves mean curvature flow

$$\left(\frac{\partial F}{\partial t}\right)^{\perp} = \vec{H} = -H \cdot \nu \qquad H = \lambda_1 + \dots + \lambda_n$$

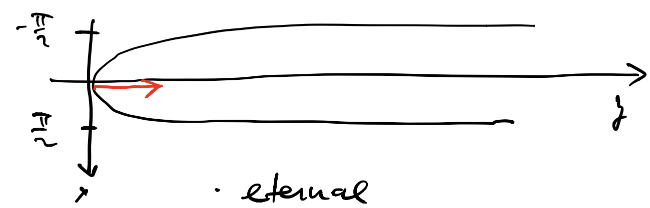
$$F(\cdot, 0) = F_0$$

**Examples:**

- shrinking spheres $S^n_{R(t)}(p)$, $R(t) = \sqrt{R_0^2 - 2nt}$
- —"— cylinders $S^k_{R(t)}(p) \times \mathbb{R}^{n-k}$, $R(t) = \sqrt{R_0^2 - 2kt}$
- grim reaper

  $y(x) = -\ln \cos(x)$

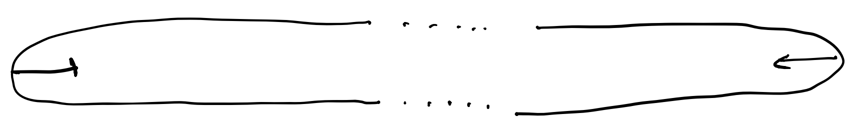
  $|x| < \pi/2$




  · eternal

  · translating

- paper clip:



  · ancient

**Thm** (Gage–Hamilton '86, Grayson '87): curve shortening flow contracts simple, closed curves in $\mathbb{R}^2$ to "round points" in finite time.

Thm (Huisken '84) MCF contracts closed, convex hypersurfaces in $\mathbb{R}^{n+1}$ in finite time to 'round points'.

Basic properties:

- for a flow with normal speed $-f \cdot \nu$, we have

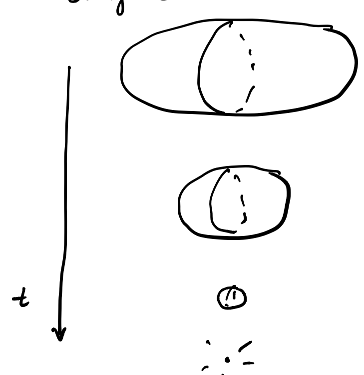
$$\frac{d}{dt}|M_t| = \frac{d}{dt}\int 1 \, d\mu_t = -\int f \cdot H \, d\mu_t$$

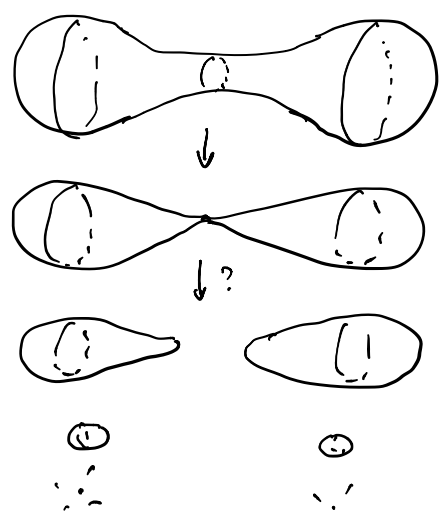
$$\Rightarrow \quad \frac{d}{dt}|M_t| = -\int H^2 \, d\mu_t \quad \text{for MCF} \quad (L^2\text{-gradient flow})$$

- smooth short-time existence
- avoidance principle: assume $(M_t^1)_{t\in[0,T)}$, $(M_t^2)_{t\in[0,T)}$ two MCFs s.t. $M_0^1 \cap M_0^2 = \emptyset$

$$\Rightarrow M_t^1 \cap M_t^2 = \emptyset \quad \forall t \in [0,T)$$

- embeddedness preserved
- finite existence time.
- singularities



Weak compactness for submanifolds

Recall the natural estimate

$$\int_0^T \int_{M_t} |H|^2 \, d\mu_t \, dt \leq |M_0|$$

Q: compactness for submanifolds under weak mean curvature bounds?

I.e. $(M_i)$ sequence of $m$-submanifolds of $\mathbb{R}^N$,

$\mathcal{H}^m(M_i \cap K) \leq C(K) \quad \forall i, \forall K \subset\subset \mathbb{R}^N$. Assume

$$\int_{M_i \cap K} |H|^2 \, d\mu_i \leq C(K) \qquad \forall i$$

Can we understand in what sense

$M_i \longrightarrow M_\infty$ ? What structure does $M_\infty$ have.

Varifolds: $M \subset \mathbb{R}^N$ $m$-submanifold, get a Radon measure by

$$\mu_M(S) = \mathcal{H}^m(M \cap S)$$

$$\int f \, d\mu_M := \int_M f \, d\mathcal{H}^m \qquad f \in C_c^0(\mathbb{R}^N)$$

So for the $M_i$ as above we have (up to a subsequence)

$$\mu_{M_i} \longrightarrow \mu$$

But: quite coarse, so wish to refine it.

Note: $M$ defines a Radon measure on $\mathbb{R}^N \times G(m,N)$, the Grassmann bundle of $m$-dim. subspaces of $\mathbb{R}^N$

Defn: $V_M$

$$\int f \, dV_M = \int_M f(x, \mathrm{Tan}(M,x)) \, d\mathcal{H}^m$$

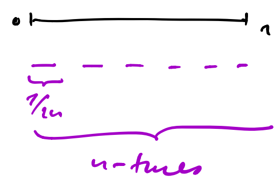
for $f \in C_c^0(\mathbb{R}^N \times G(m,N))$. Note

$$V_M(S) = \mathcal{H}^m(\{x \in M : (x, \mathrm{Tan}(M,x)) \in S\})$$

Can assume $V_{M_i} \longrightarrow V$. Note $\pi_* V_M = \mu_M$

<u>Def</u>: An $m$-dim. varifold is a Radon measure on $\mathbb{R}^N \times G(m,N)$.

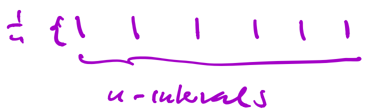
<u>Examples of varifold convergence</u>: work in $\mathbb{R}^2$

i) $M_n$ 

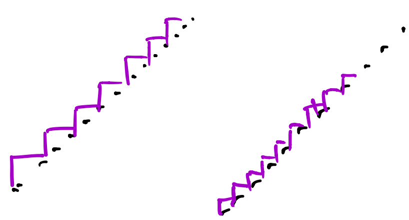
$\subset \mathbb{R}^2$

$\mu_{M_n} \longrightarrow \frac{1}{2} \mu_{[0,1]}$

$V_{M_n} \longrightarrow \frac{1}{2} V_{[0,1]}$

ii) 


$\mu_{M_n} \longrightarrow \mu_{[0,1]}$ ; $V_{M_n} \not\longrightarrow V_{[0,1]}$

iii) $M_n$ 

$\mu_{M_n} \longrightarrow \sqrt{2}\, \mu_{\mathrm{diag}}$

but $\exists\ \alpha > 0$ s.t.

$V_{M_n} \longrightarrow \alpha V_{\mathrm{diag}}$

## <u>Integral Varifolds</u>

<u>Lemma</u>: Let $M, M'$ be $m$-dim. $C^1$-submanifolds of $\mathbb{R}^N$. Let

$$Z = \{x \in M \cap M' : \mathrm{Tan}(M,x) \neq \mathrm{Tan}(M',x)\}.$$

Then $\mathcal{H}^m(Z) = 0$

Pf: If $M, M'$ are hypersurfaces, then $Z$ is a $(m-1)$-dim $C^1$-submanifold by transversality. In higher codimension, one may show that $Z \subset M \cap M'$ is contained in an $(m-1)$-dim $C^1$-submanifold, by projection to a lower dim subspace (exercise) $\square$.

Cor: Assume $S \subset \bigcup_i M_i$ and $S \subset \bigcup_i M_i'$ for $M_i, M_i'$ $m$-dim submanifolds of $\mathbb{R}^N$. Define $T$ on $S$ by

$$T(x) = \operatorname{Tan}(M_i, x)$$

where $i$ is the first $i$ st. $x \in M_i$, i.e. $x \in M_i \setminus \bigcup_{j<i} M_j$

Define $T'$ similarly. Then

$$T(x) = T'(x)$$

for a.e. $x \in S$

Thus for $S$ a Borel subset of a $C^1$ submanifold $M \subset \mathbb{R}^N$ we define $V_S$ to be the varifold given by

$$\int f \, dV_S = \int_S f(x, \operatorname{Tan}(M, x)) \, d\mathcal{H}^m(x)$$

This does not depend on the choice of $M$ by the previous discussion.

Def: An integral $m$-varifold is a varifold which can be written as

$$V = \sum_{i=1}^{\infty} V_{S_i}$$

## First variation of a varifold

**Thm (divergence thm)** Assume $M$ is a $C^2$-submanifold of $\mathbb{R}^N$ and $X \in C^1_c(\mathbb{R}^N; \mathbb{R}^N)$. Let

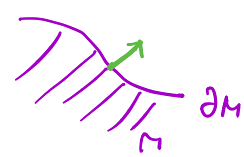
$$(\mathrm{div}_M(X))(x) = \sum_{i=1}^{m} \langle \nabla_{e_i} X(x), e_i \rangle$$

for $e_1, \dots, e_m$ an ON-Basis of $T(M,x)$. Then

$$\int_M \mathrm{div}_M(X) = \int_M \mathrm{div}_M(X^\perp) + \int_M \mathrm{div}\,(X^T)$$

$$= -\int_M \langle X, \vec{H} \rangle \, d\mathcal{H}^m + \int_{\partial M} \langle X, \vec{u} \rangle \, d\mathcal{H}^{m-1}$$

where $\vec{u}$ is the exterior unit conormal to $\partial M$:




Note: LHS makes sense for $M$ merely in $C^1$.

If there is a distributional vector field $\vec{H}$ making this true, we say $\vec{H}$ is the weak mean curvature.

For $V$ an (integral) $m$-varifold, define the first variation of $V$ by

$$\delta V(X) = \int \mathrm{div}_T(X) \; dV(x,T)$$

$$= \int \mathrm{div}_{T(V,x)}(X) \; d\mu_V(x)$$

Rem: $X \in C^1_c(\mathbb{R}^N;\mathbb{R}^N)$, $(\phi_t)_{-\varepsilon<t<\varepsilon}$ s.t. $\phi_0 = \mathrm{id}$, $\frac{d}{dt}\phi_t = X \circ \phi_t$, then

$$\frac{d}{dt}\Big( ((\phi_t)_\# V)(K)\Big)\Big|_{t=0} = \delta V(X)$$

$$\operatorname{supp}(X) \subset\subset K \subset\subset \mathbb{R}^N$$

Note: If $V_i \longrightarrow V$, then $\delta V_i(X) \longrightarrow \delta V(X)$.

Assume we have local bounds on the first variation;

i.e.

$$|\delta V(X)| \leq C_K \|X\|_\infty$$

for $\operatorname{supp}(X) \subset K \subset\subset \mathbb{R}^N$

Riesz repr. thm: $\exists$ Radon measure $\lambda$ and $\lambda$-measurable unit vectorfield $\Lambda$ s.t.

$$\delta V(X) = \int \langle X, \Lambda\rangle \, d\lambda$$

Decompose $\lambda$ w.r.t $\mu_V$, then is $\lambda_{ac} << \mu_V$ and $\lambda_{sing}$ s.t.

$$\delta V(X) = \int \langle X, \Lambda\rangle \, d\lambda_{ac} + \int \langle X, \Lambda\rangle \, d\lambda_{sing}$$

$$= \int \langle X, \underbrace{\Lambda \frac{d\lambda_{ac}}{d\mu_V}}_{=:-\vec{H}} \rangle \, d\mu_V + \int \langle X, \underbrace{\Lambda}_{=:\vec{u}} \rangle \, d\lambda_{sing}$$

$$\Longrightarrow \quad \delta V(X) = -\int \langle X, \vec{H}\rangle \, d\mu_V + \int \langle X, \vec{u}\rangle \, d\lambda_{sing}$$

**Thm** (Allard's compactness theorem)

Suppose $V_i \to V$ is a sequence of integral varifolds converging weakly to a varifold $V$. If the $V_i$ have locally uniformly bounded first variation; i.e. for $K \subset\subset \mathbb{R}^n$ there is $C_K$ (indep. of $i$) s.t. $\forall X \in C^1_c(\mathbb{R}^N;\mathbb{R}^N)$ with $\operatorname{supp}(X) \subset K$ we have

$$|\delta V_i(X)| \le C_K \|X\|_\infty$$

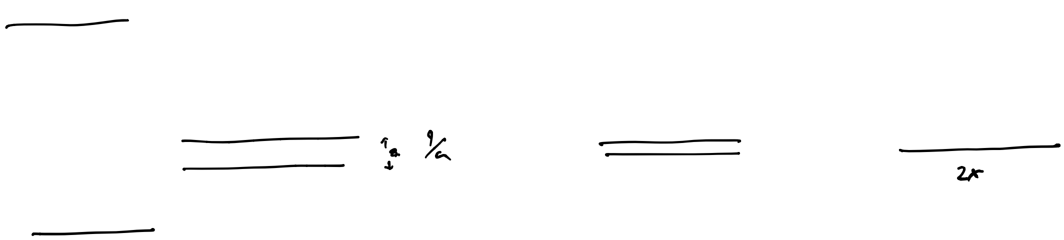
then $V$ is also an integral varifold.

Rem: i) We trivially obtain $|\delta V(X)| \le C_K \|X\|_\infty$

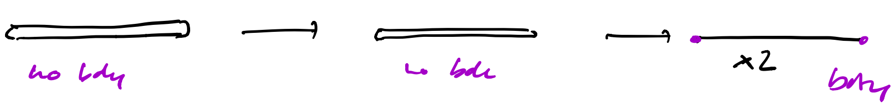
ii) A sequence of smooth ($C^2$)-submanifolds satisfies the hypotheses of Allard's thm iff

$$\int_K |H_i| \, d\mu_{M_i} + \int_K d\sigma_i \le C_K$$

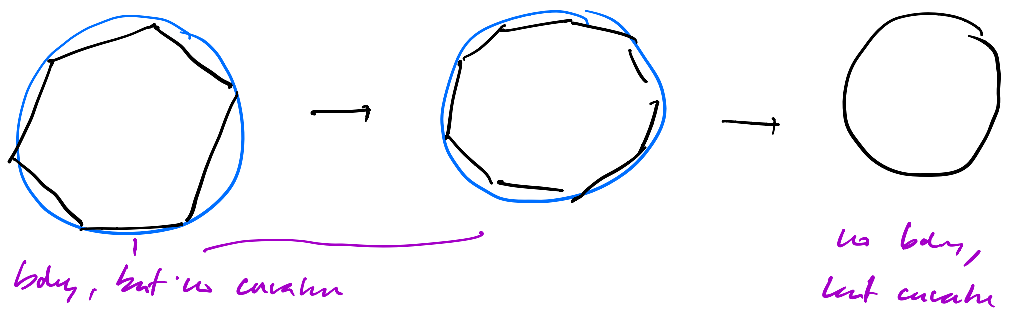
where the $d\sigma_i$ is the boundary measure of the $M_i$.




iii) The quantities '$|H|$' and '$d\sigma$' can be mixed up in the limit.

a) 

b)




## Brakke flows

**Def:** An ($n$-dim) integral Brakke flow in $\mathbb{R}^{n+1}$ is a 1-parameter family of Radon measures $(\mu(t))_{t\in I}$ over an interval $I \subset \mathbb{R}$, s.t.

(1) for almost all $t \in I$, there exists an integral $n$-varifold $V(t)$ with $\mu(t) = \mu_{V(t)}$ s.t. $V(t)$ has locally bounded first variation and has weak mean curvature $\vec{H}$ orthogonal to $T(V(t),\cdot)$ almost every.

(2) For a bounded interval $[t_1,t_2] \subset I$ and any compact set $K \subset \mathbb{R}^{n+1}$,

$$\int_{t_1}^{t_2}\int_K \left(1+|\vec{H}|^2\right) d\mu(t)\, dt < \infty$$

(3) If $[t_1,t_2] \subset I$ and $f \in C^1_c(\mathbb{R}^{n+1} \times [t_1,t_2])$ has $f \geq 0$, then

$$\int f(\cdot,t_2)\, d\mu(t_2) - \int f(\cdot,t_1)\, d\mu(t_1) \leq \int_{t_1}^{t_2}\int \left(-|\vec{H}|^2 f + \langle \vec{H}, \nabla f\rangle + \frac{\partial f}{\partial t}\right) d\mu(t)\, dt$$

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Rem: If $(M_t)_{t\in[t_1,t_2]}$ is a smooth MCF

$$\frac{d}{dt}\int_{M_t} f\,dA = \int_{M_t}\left(-|H|^2 f + \langle \nabla f, \vec{H}\rangle + \frac{\partial f}{\partial t}\right) dA$$

So why do we only require the inequality in (3)?

Ex assume $f_i \in W^{1,2}_0(\Omega)$, $\int |Df_i|^2\,dx \le C$

Can assume $f_i \to f_\infty$

$$\int |Df_\infty|^2\,dx \le \liminf_{i\to\infty}\int |Df_i|^2$$