Introduction to Brakke flows II

Today:
1. Brakke flows: basic properties
2. Compactness
3. Existence via elliptic regularisation

Recall

Def: An (n-dim) integral Brakke flow in $\mathbb{R}^{n+1}$ is a 1-parameter family of Radon measures $(\mu(t))_{t\in I}$ over an interval $I \subset \mathbb{R}$, s.t.

(1) for almost all $t \in I$, there exists an integral n-varifold $V(t)$ with $\mu(t) = \mu_{V(t)}$ s.t. $V(t)$ has locally bounded first variation, which is absolutely continuous w.r.t. $\mu(t)$ and has weak mean curvature $\vec{H}$ orthogonal to $T(V(t), \cdot)$ almost everywhere.

(2) For a bounded interval $[t_1, t_2] \subset I$ and any compact set $K \subset \mathbb{R}^{n+1}$

$$\int_{t_1}^{t_2}\int_K \left(1 + |\vec{H}|^2\right) d\mu(t)\, dt < \infty$$

(3) If $[t_1, t_2] \subset I$ and $f \in C^1_c(\mathbb{R}^{n+1} \times [t_1, t_2])$ has $f \geq 0$, then

$$\int f(\cdot, t_2)\, d\mu(t_2) - \int f(\cdot, t_1)\, d\mu(t_1)$$
$$\leq \int_{t_1}^{t_2}\int \left(-|\vec{H}|^2 f + \langle \vec{H}, Df\rangle + \frac{\partial f}{\partial t}\right) d\mu(t)\, dt$$

Rem: (i) If $(M_t)_{t\in I}$ is a smooth MCF, then

$$\frac{d}{dt}\int_{M_t} f\, dA = \int_{M_t} \left(-|\vec{H}|^2 f + \langle \nabla f, \vec{H}\rangle + \frac{\partial f}{\partial t}\right) dA.$$

(ii) Assume that $(M_t)_{t\in I}$ is a smooth family of hypersurfaces, satisfying (3), then one can show that $(M_t)_{t\in I}$ solves MCF smoothly.

Thm: $(\mu_t)_{t\in I}$ an integral (n-dim) Brakke flow on $\mathbb{R}^{n+1}$. Let $\phi = (r^2 - |x|^2 - 2nt)^+$. Then

$$\int \phi^4 \, d\mu_t$$

is decreasing in t

Pf: Let $f = \frac{1}{4}\phi^4$, then $Df = \phi^3 D\phi$ and

$$\operatorname{div}_M (Df) = 3\phi^2 |\nabla^T \phi|^2 + \phi^3 \operatorname{div}(D\phi) \geq -2n\phi^3$$

Moreover $$\frac{\partial f}{\partial t} = \phi^3 \frac{\partial \phi}{\partial t} = -2n\phi^3$$

Apply the Brakke flow definition to f:

$$\int f \, d\mu_b - \int f \, d\mu_a \leq \int_a^b \int \left(-|H|^2 + \langle \vec{H}, Df\rangle + \frac{\partial f}{\partial t}\right) d\mu_t \, dt$$

$$\leq \int_a^b \int \left(-\operatorname{div}_M(Df) + \frac{\partial f}{\partial t}\right) d\mu_t \, dt \leq 0$$

$\square$

Cor: For an integral (n-dim) Brakke flow (μ_t) on $\mathbb{R}^{n+1}$, defined on $[a,b]$, $K \subset \mathbb{R}^{n+1}$ compact, we have uniform mass bounds, i.e. there is C_K, independent of t, s.t.

$$\mu_t(K) \leq C_K < \infty \qquad t \in [a,b]$$

Thm: An integral Brakke flow satisfies for $K \subset\subset \mathbb{R}^{n+1}$

$$\int_a^b \int_K |\vec{H}|^2 \, d\mu_t \, dt \leq C_K' (1+b-a)$$

pf: $\phi \in C_c^2(\mathbb{R}^{n+1})$, $\phi \geq 0$, time independent.

Recall $\frac{|D\phi|^2}{\phi} \leq C(|D^2\phi|)$ (Exercise)

Since
$$\langle D\phi, \vec{H} \rangle \leq \frac{1}{2} \frac{|D\phi|^2}{\phi} + \frac{1}{2} \phi |\vec{H}|^2$$

we have
$$\int \phi \, d\mu_b - \int \phi \, d\mu_a \geq \int_a^b \int \left(\phi |\vec{H}|^2 - \langle D\phi, \vec{H} \rangle \right) d\mu_t \, dt$$

$$\geq \int_a^b \int \left(\tfrac{1}{2} \phi |H|^2 - \tfrac{1}{2} \frac{|D\phi|^2}{\phi} \right) d\mu_t \, dt$$

$\Rightarrow$
$$\int_a^b \int \tfrac{1}{2} \phi |H|^2 d\mu_t \, dt \leq \int \phi \, d\mu_a - \int \phi \, d\mu_b + \frac{1}{2} \int_a^b \int \frac{|D\phi|^2}{\phi} d\mu_t \, dt$$

$$\leq \int \phi \, d\mu_a - \int \phi \, d\mu_b + C(\phi) \int_a^b \int \chi_{\{\phi \neq 0\}} d\mu_t \, dt$$

$$\leq C(\phi) \, C_K (1+b-a)$$

where $\{\phi \neq 0\} \subset K$. $\square$

Thm: An integral (n-dim) Brakke flow satisfies

$$\lim_{\tau \nearrow t} \mu_\tau \geq \mu_t \geq \lim_{\tau \searrow t} \mu_\tau \quad \text{i.e.}$$

for $\phi \in C_c^0(\mathbb{R}^{n+1})$ with $\phi \geq 0$ we have

$$\lim_{\tau \nearrow t} \int \phi \, d\mu_\tau \geq \int \phi \, d\mu_t \geq \lim_{\tau \searrow t} \int \phi \, d\mu_\tau$$

Pf: Assume first $\phi \in C_c^2(\mathbb{R}^{n+1})$, $\phi \geq 0$. [general case follows by approximation]. Then

$$\int \phi \, d\mu_d - \int \phi \, d\mu_c \leq \int_c^d \int (-\phi |H|^2 + \langle \vec{H}, D\phi \rangle) \, d\mu_t \, dt$$

$$\leq \int_c^d \int \frac{1}{2} \frac{|D\phi|^2}{\phi} \, d\mu_t \, dt$$

$$\leq C(\phi) \, c_{\operatorname{supp}\phi} \, (d-c)$$

$\Longrightarrow$ $f(t) := \int \phi \, d\mu_t - C(\phi) \, c_{\operatorname{supp}\phi} \, t$ is decreasing in t

$\Longrightarrow$ $f(t^-) \geq f(t) \geq f(t^+)$

this finishes the proof, as the linear part of f is cont. $\square$

Cor: For an integral (n-dim) Brakke flow in $\mathbb{R}^{n+1}$ and $\phi \in C_c^2(\mathbb{R}^{n+1})$, $\phi \geq 0$, the map

$$t \longmapsto \int \phi \, d\mu_t - C(\phi) \, c_{\operatorname{supp}\phi} \cdot t$$

is decreasing in t.

Compactness:

Thm: Suppose $[a,b] \ni t \longmapsto \mu_t^i$ is a sequence of integral Brakke flows on $\mathbb{R}^{n+1}$. Assume the bounds on area are locally uniform, i.e.

$$\sup_i \sup_{t\in[a,b]} \mu_t^i(K) \le C_K < \infty \qquad \forall K \subset\subset \mathbb{R}^{n+1}$$

Then after passing to a subsequence

(1) $\mu_t^i \longrightarrow \mu_t \qquad \forall t \in [a,b]$

(2) $[a,b] \ni t \longmapsto \mu_t$ is an integral Brakke flow

(3) for a.e. $t \in [a,b]$, after passing to a further subsequence which depends on t, the associated varifolds converge nicely: $V_t^i \longrightarrow V_t$.

Pf: Choose $\phi \in C_c^2(\mathbb{R}^{n+1})$, $\phi \ge 0$. Recall

$$L_i^\phi(t) = \int \phi \, d\mu_t^i - C(\phi) C_{\operatorname{supp}\phi} t$$

is a sequence of uniformly bdd, decreasing fcts of t. Passing to a subsequence, depending on ϕ, we have that $L_i^\phi(t)$ converges pointwise to a decreasing fct. $L(t)$ (Exercise)

Hence
$$\int \phi \, d\mu_t^i$$

has a limit for all $t \in [a,b]$. Choose a countable dense subset $\mathcal{S} \subset C_c^0(\mathbb{R}^{n+1}; \mathbb{R}^+)$ of functions in $C_c^2(\mathbb{R}^{n+1}; \mathbb{R}^+)$. Repeating the above process for each $\phi \in \mathcal{S}$ (choosing a diagonal subsequence), we see that there is a subsequence i_i s.t. for all $\phi \in \mathcal{S}$

$$\int \phi \, d\mu^i_t$$

has a limit, for all $t \in [a,b]$. By density, this extends to all of $C^0_c(\mathbb{R}^{n+1}; \mathbb{R}^+)$.

Since the limits are unique, we have that

$$\mu^i_t \longrightarrow \mu_t$$

for a family of Radon measures $[a,b] \ni t \longmapsto \mu_t$.

Aim: show that this is a Brakke flow and show strengthened convergence.

Now, replace $\mathbb{R}^{n+1}$ by $U \subset\subset \mathbb{R}^{n+1}$ for simplicity, U open.

Thus, we may assume that $\mu^i_t(U) \leq C < \infty$ independent of i and t. We have also shown that we can assume that

$$\int_a^b \int |H|^2 \, d\mu^i_t \, dt \leq D < \infty$$

independent of i. Let $[c,d] \subset [a,b]$. Then

$$\int \phi \, d\mu^i_c - \int \phi \, d\mu^i_d \geq \int_c^d \int \left(\phi |\vec{H}_i|^2 - \langle D\phi, \vec{H}_i \rangle - \frac{\partial \phi}{\partial t} \right) d\mu^i_t dt$$

$\Longrightarrow$

$$\int \phi \, d\mu_c^i - \int \phi \, d\mu_d^i + \varepsilon D + \int_c^d \int \frac{1}{2\varepsilon} |D\phi|^2 \, d\mu_t^i \, dt$$

$$\geq \int_c^d \int \left(\phi |\vec{H}|^2 - \langle D\phi, \vec{H} \rangle + \varepsilon |H_i|^2 + \frac{1}{2\varepsilon} |D\phi|^2 - \frac{\partial \phi}{\partial t} \right) d\mu_t^i \, dt$$

Since $\langle D\phi, \vec{H}_i \rangle \leq \frac{1}{2\varepsilon} |D\phi|^2 + \frac{\varepsilon}{2} |\vec{H}_i|^2$, so

$$\phi |H|^2 - \langle D\phi, \vec{H} \rangle + \varepsilon |H_i|^2 + \frac{1}{2\varepsilon} |D\phi|^2 \geq \frac{\varepsilon}{2} |H_i|^2 \geq 0$$

Pass to a limit in i and use Fatou's lemma to see that

$$\int \phi \, d\mu_c - \int \phi \, d\mu_d + \varepsilon D + \int_c^d \int \frac{1}{2\varepsilon} |D\phi|^2 \, d\mu_t \, dt$$

$$\geq \int_c^d \liminf_i \int \left(\phi |H_i|^2 - \langle D\phi, \vec{H}_i \rangle + \varepsilon |H_i|^2 + \frac{1}{2\varepsilon} |D\phi|^2 \right) d\mu_t \, dt$$

$$- \int_c^d \int \frac{\partial \phi}{\partial t} \, d\mu_t \, dt$$

$\Rightarrow$ for a.e. $t \in [c,d]$ we have

$$C(t) := \liminf_i \int \left(\phi |H_i|^2 - \langle D\phi, \vec{H} \rangle + \varepsilon |H_i|^2 + \frac{1}{2\varepsilon} |D\phi|^2 \right) d\mu_t < \infty$$

Pass to a subsequence (depending on t!) so this becomes a limit, rather than a lim inf. Since the integrand is bounded below by $\frac{\varepsilon}{2} |H_i|^2$, we see that

the μ_t^i are integral varifold with mean curvature uniformly in $\mathcal{L}^2(\mu_t)$. Hence we can apply Allard's theorem (slightly strengthened version, see notes) to pass to a subsequence s.t. $V_t^i \longrightarrow V_t$

where V_t is an integral varifold with $H \in \mathcal{L}^2(d\mu_V)$.

In particular

$$\int \langle \vec{H}_i, X\rangle \, d\mu_{V_t^i} \longrightarrow \int \langle \vec{H}, X\rangle \, d\mu_{V_t}$$

for $X \in C_c(\mathcal{U}; \mathbb{R}^{n+1})$ a cont. vectorfield.

Note: For a.e. t. $V(t)$ is well defined independent of the subsequence dep. on t: this is because an integral varifold V is uniquely defined by its associated measure μ_V.

Returning to $C(t)$, each term converges to what we expect, except for $|H_i|^2$ terms, which might drop in general (by weak convergence, just use duality)

Hence we see that

$$C(t) \geq \int \left(\phi \, |\vec{H}|^2 - \langle D\phi, \vec{H}\rangle + \varepsilon |H|^2 + \frac{1}{2\varepsilon} |D\phi|^2 \right) d\mu_t$$

Cancel the terms with $\frac{1}{2\varepsilon} |D\phi|^2$ on LHS and RHS and letting $\varepsilon \to 0$, we see that (μ_t) is a Brakke flow.

$\square$.

Existence by elliptic regularisation

compare Ilmanen '94 : 'Elliptic regularisation and partial regularity for motion by mean curvature'. Notices of the AMS.

Thm: Let z denote the height fct in $\mathbb{R}^{n+2} = \mathbb{R}^{n+1} \times \mathbb{R}$ and $\vec{e}$ the upward pointing unit vector. Then $M \subset \mathbb{R}^{n+2}$ is a critical point of

$$\int_M e^{-\lambda z} dA$$

iff $\qquad t \longmapsto M - \lambda t \vec{e} \qquad$ solves MCF

Pf: Let $s \mapsto M_s$ be a variation of M with velocity X, then

$$\frac{d}{ds} \int_{M_s} e^{-\lambda s} dA \Big|_{s=0} = \int \langle -\vec{H} + D^\perp(-\lambda z), X \rangle e^{-\lambda z} dA$$

$$= \int \langle -\vec{H} - \lambda e^\perp, X \rangle e^{-\lambda z} dA$$

On the other hand $t \longmapsto M - \lambda t \vec{e}$ has velocity $-\lambda \vec{e}$, and thus normal velocity $-\lambda e^\perp$. Comparing yields the statement. $\square$

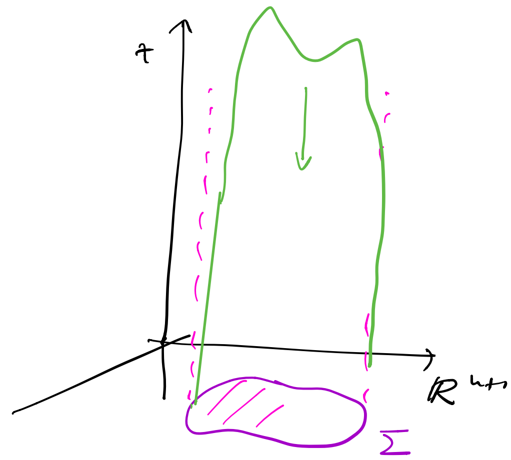
Setup:

- $\Sigma^n \subset \mathbb{R}^{n+1} \times \{0\} \subset \mathbb{R}^{n+1} \times \mathbb{R}$, closed, let $M_\lambda \subset \mathbb{R}^{n+2}$ minimise

$$\int e^{-\lambda z} dA \quad \text{subject to } \partial M_\lambda = \Sigma$$

- Can show M_λ exist and in nice situations (i.e. for hypersurfaces) are smooth away from a small sing. set.
- Thus we get $t \longmapsto \underbrace{M_\lambda - \lambda t \vec{e}}_{M_t^\lambda}$ is a 'smooth'

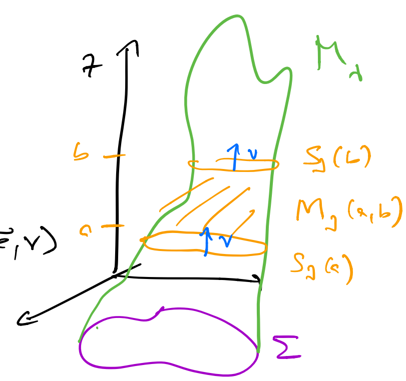
mean curvature flow.



<u>Goal:</u> find $\lambda \to \infty$, and show that $(M^\lambda_t)_{t\in\mathbb{R}^+}$ converge to a limit Brakke flow $(\mu_t)_{t\geq 0}$ on $\mathbb{R}^{n+2}$, which is translation invariant, i.e.,
$\mu_t = \Sigma_t \times \mathbb{R}$ for Σ_t an n-dim Brakke flow in $\mathbb{R}^{n+1}$ with $\Sigma_0 = \Sigma$.

<u>Uniform local area estimates:</u>

Let $M_\lambda(a,b) = M_\lambda \cap \{a < z < b\}$, $S_\lambda(z_0) = M_\lambda \cap \{z = z_0\}$
$0 \leq a < b$.
ν upward pointing unit normal vector to $\partial M_\lambda(a,b)$ in $M_\lambda(a,b)$



$$0 = \int_{M_\lambda(a,b)} \mathrm{div}_{M_\lambda}(\vec{e}) = \int_{M_\lambda(a,b)} -\langle \vec{H}, \vec{e}\rangle + \int_{S_\lambda(b)} \langle \vec{e}, \nu\rangle - \int_{S_\lambda(a)} \langle \vec{e}, \nu\rangle$$

$$= \int_{M_\lambda} \lambda |e^\perp|^2 + \int_{S_\lambda(b)} |\vec{e}^T| - \int_{S_\lambda(a)} |\vec{e}^T|$$

$$\Rightarrow \quad \int_{M_\lambda(a,b)} \lambda |\vec{e}^{\perp}|^2 + \int_{S_\lambda(b)} |\vec{e}^{T}| = \int_{S_\lambda(a)} |\vec{e}^{T}|$$

$$\Rightarrow \quad z \longmapsto \int_{S_\lambda(z)} |\vec{e}^{T}| \quad \text{decreasing}$$

Now

$$\text{area}\,(M_\lambda(a,b)) = \int_{M_\lambda(a,b)} |e^{\perp}|^2 + |e^{T}|^2 \leq \frac{1}{\lambda}\int_{S_\lambda(a)} |\vec{e}^{T}| + \int_{M_\lambda(a,b)} |e^{T}|^2$$

$$= \frac{1}{\lambda}\int_{S_\lambda(a)} |\vec{e}^{T}| + \int_{z=a}^{z=b} \int_{S_\lambda(z)} |\vec{e}^{T}|\, d\mathcal{H}^{n}\, dz$$

$$\leq (\lambda^{-1} + (b-a)) \int_{S_\lambda(a)} |e^{T}| \leq (\lambda^{-1} + (b-a))\, \text{area}\,(\Sigma)$$

Next time:

- compactness: Ilmanen Brakke flow
- slow: translation invariance.

- monotonicity formula
- entropy
- tangent flows.

$$f$$

$$\int_{M^n} g |\nabla^T f| = \int_{-\infty}^{+\infty} \int_{\{f=z\}\cap M} g \, d\mathcal{H}^{n-1}$$