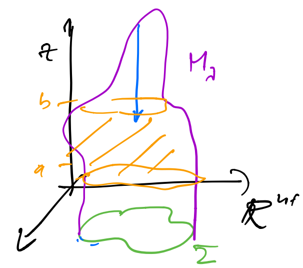
Introduction to Brakke flows III

Today

① Existence via elliptic regularisation
② Monotonicity formula
③ Entropy
④ Tangent flows



Elliptic regularisation

Recall:
- $\Sigma^n \subset \mathbb{R}^{n+1} \times \{0\} \subset \mathbb{R}^{n+1} \times_{\vec{e}, z} \mathbb{R}$ closed
- $M_\lambda^{n+1} \subset \mathbb{R}^{n+2}$ minimizing $\int e^{-\lambda z} \, dA$ subject to $\partial M_\lambda = \Sigma$
- get $\mathbb{R}^+ \ni t \longmapsto \underbrace{M_\lambda - \lambda t \vec{e}}_{M_t^\lambda}$ translating mean curvature flows

Area bounds: Have shown

$$\text{area}\,(M_\lambda \cap \{a < z < b\}) \leq (\lambda^{-1} + b - a)\,\text{area}\,(\Sigma) \qquad 0 \leq a < b$$

Can thus by compactness choose $\lambda_i \to \infty$ s.t.

$$\mathbb{R}^+ \ni t \longmapsto \mathcal{H}^{n+1} \llcorner M_t^\lambda =: \mu_t^\lambda \quad \text{converges as } \lambda_i \to \infty$$

to a Brakke flow

$$\mathbb{R}^+ \ni t \longmapsto \mu_t \text{ in } \mathbb{R}^{n+1} \times \mathbb{R}^+$$

Want to show:

(a) the flow is translation invariant, i.e.

$\mu_t = \Sigma_t \times \mathbb{R}^+$ for Σ_t a Brakke flow on $\mathbb{R}^{n+1}$

(b) it has initial condition $\Sigma \times \mathbb{R}^+$

Translation invariance

$\phi \in C_c^2(\mathbb{R}^{n+1} \times \mathbb{R}^+ ; \mathbb{R}^+)$. Let $\phi^\tau(x,z) := \phi(x, z-\tau)$

Recall $\quad (M_t^{\lambda_i})_{t \geq 0} \longrightarrow (\mu_t)_{t \geq 0}$

Note $\quad \mu_t^\lambda(\phi^\tau) \longrightarrow \mu_{t+\frac{\tau}{\lambda}}^\lambda(\phi)$

Recall $\exists$ constant c_ϕ independent of λ s.t.

$$t \longmapsto \mu_t^\lambda(\phi) - c_\phi t \quad \searrow$$

hence for $t < s$, for λ large enough s.t. $t < t + \tau/\lambda < s$ we have

$$\mu_t^\lambda(\phi) - c_\phi t \geq \underbrace{\mu_{t+\tau/\lambda}^\lambda(\phi)}_{=\mu_t^\lambda(\phi^\tau)} - c_\phi(t + \tau/\lambda) \geq \mu_s^\lambda(\phi) - c_\phi s$$

letting $\lambda_i \to \infty$ this implies

$$\mu_t(\phi) - c_\phi t \geq \mu_t(\phi^\tau) - c_\phi t \geq \mu_s(\phi) - c_\phi s$$

$\overset{s \downarrow t}{\Longrightarrow}$

$$\mu_t(\phi) \geq \mu_t(\phi^\tau) \geq \mu_{t^+}(\phi) \qquad \forall t$$

Moreover (using density), for all but countably many t, $t \mapsto \mu_t$ is continuous at t

$$\Rightarrow \mu_t = \Sigma_t \times \mathbb{R}^+ \qquad \text{for all but countably many } t.$$

(then fill in the blanks by limits from above (or below).

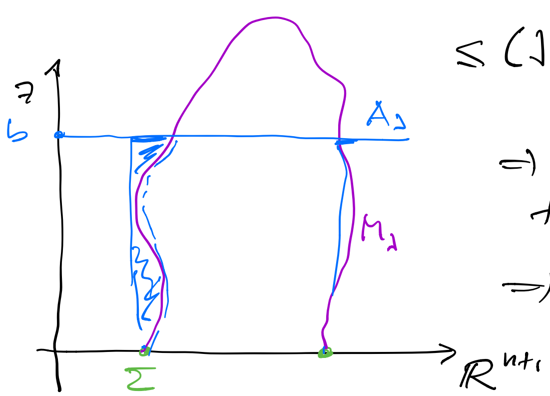
Initial condition: $\mu_0 = \mathcal{H}^{n+1} \llcorner \Sigma \times \mathbb{R}^+$

Recall: Flat norm $\mathcal{F}(\cdot)$ for A, B m-dim cycles (i.e. m-dim surfaces S, st. $\partial S \neq \phi$), the flat norm $\mathcal{F}(A-B)$ is the infimum of the area of $(m+1)$-chains spanning $A-B$.

Let Π_b be the projection on $\mathbb{R}^n \times \{b\}$ and let $A_\lambda = \Pi_b(M_\lambda(0,b))$. Then

$$\text{area}(A_\lambda) \leq \int_{M_\lambda(0,b)} |\vec{e}^{n+1}| \leq \left(\int_{M_\lambda(0,b)} |\vec{e}^{n+1}|^2\right)^{\frac{1}{2}} \underbrace{(\text{area}(M_\lambda(0,b))}_{\leq c})^{\frac{1}{2}}$$

$$\leq (\lambda^{-1}\text{area}(\Sigma))^{\frac{1}{2}} \cdot C \longrightarrow 0$$



$\Rightarrow$ area in blue region goes to zero

$\Rightarrow \mathcal{F}(M_\lambda(0,b) + A_\lambda - \Sigma \times (0,b)) \longrightarrow 0$

Prop: Assume $\mathcal{F}(T_i - T) \to 0$ (i.e. $T_i \longrightarrow T$ as currents in flat norm)

Then $\quad \mathbb{M}(T) \leq \liminf (\mathbb{M}(T_i))$ (lower semi-continuity of mass under flat convergence)

even more $\quad \mu_T \leq \liminf \mu_{T_i}$

This implies that if the total mass converges, then measures have to converge

$$\begin{array}{ccc} \mu_{M_i} & \longrightarrow & \mu_{\Sigma_0 \times \mathbb{R}^+} \\ \| & & \\ \mu_0^i & \longrightarrow & \mu_0 \end{array}$$

<u>Monotonicity formula</u>

We say a measure μ_0 has bounded n-dim area ratios if

$$\sup_{\substack{x\in\mathbb{R}^{n+1}\\ r>0}} \frac{\mu_0(B_r(x))}{\omega_n r^n} \le D < \infty \qquad (*)$$

<u>Lemma</u>: Assume $(\mu_t)_{t\geq 0}$ is an (n-dim) integral Brakke flow on $\mathbb{R}^{n+1}$ satisfying $(*)$. Then for $t\in[0,\frac{r^2}{4n}]$

$$\sup_{x\in\mathbb{R}^{n+1}} \mu_t(B_r(x)) \le 2^{n+2} D r^n \qquad (**)$$

<u>Pf</u>: This follows from the monotonicity of

$$t \longmapsto \int (r^2-|x|^2-2nt)_+^4 \, d\mu_t$$

as discussed before. $\square$

Define backwards heat kernel, centered at $X_0=(x_0,t_0) \in\mathbb{R}^{n+1}\times\mathbb{R}$

$$\rho_{X_0}(x,t) := (4\pi(t_0-t))^{-n/2} e^{-\frac{|x-x_0|^2}{4(t_0-t)}}$$

<u>Thm</u> (Monotonicity formula, Huisken/Ilmanen/White)
Let $(\mu_t)_{t\geq 0}$ be an integral n-dim Brakke flow in $\mathbb{R}^{n+1}$, satisfying $(*)$. Then $\forall\, 0\le t_1<t_2<t_0$:

$$\int \rho_{x_0}(\cdot,t_2)\, d\mu_{t_2} + \int_{t_1}^{t_2}\int \left| \vec{H} + \frac{(x-x_0)^\perp}{2(t_0-t)} \right|^2 \rho_{x_0}(\cdot,t)\, d\mu_t\, dt$$

$$\leq \int \rho_{x_0}(\cdot,t_1)\, d\mu_{t_1}$$

<u>Proof:</u> By translation invariance we can assume $x_0 = 0$.
Let $\rho = \rho(0,t_0)$. By definition

$$\int \phi(\cdot,t_2)\, d\mu_{t_2} - \int \phi(\cdot,t_1)\, d\mu_{t_1} \leq \int_{t_1}^{t_2}\int \left(-|\vec{H}|^2 + \langle \vec{H}, \nabla\phi\rangle + \frac{\partial\phi}{\partial t}\right) d\mu_t\, dt$$

$$\phi \in C_c^1(\mathbb{R}^{n+1}\times\mathbb{R};\mathbb{R}^+)$$

whenever the inner integral on the RHS is finite (i.e. μ_t comes from an n-dim. integral varifold with first variation in $\mathcal{L}^2_{loc}(\mu_t)$), we have by the first variation formula

$$\int -|\vec{H}|^2\phi + \langle \vec{H}, \nabla\phi\rangle + \frac{\partial\phi}{\partial t}\, d\mu_t = \int -|H|^2\phi + 2\langle H, \nabla\phi\rangle + \operatorname{div}_{\mathrm{Tan}(V_{\mu_t})}(\nabla\phi) + \frac{\partial\phi}{\partial t}\, d\mu_t$$

$$= \int -\phi \left| \vec{H} - \frac{\nabla^\perp\phi}{\phi} \right|^2 + Q_{\mathrm{Tan}(V_{\mu_t})}(\phi)\, d\mu_t$$

where

$$Q_T(\phi) = \frac{|\nabla^\perp\phi|^2}{\phi} + \operatorname{div}_T(\nabla\phi) + \frac{\partial\phi}{\partial t}$$

for any n-dim subspace T of $\mathbb{R}^{n+1}$

Note $Q_T(\rho) = 0$ (Exercise)

To insert ρ into the above formula, let $\varphi = \varphi_R$ be a cut-off function s.t. $\chi_{B_R(0)} \leq \varphi \leq \chi_{B_{2R}(0)}$,

$$R|\nabla\varphi| + R^2|\nabla^2\varphi| \leq C$$

We calculate

$$Q_T(\varphi\rho) = \varphi\, Q_T(\rho) + \rho\, Q_T(\varphi) + 2\langle\nabla\varphi, \nabla\rho\rangle$$
$$\leq C\left(\frac{1}{R^2} + \frac{1}{t_0-t}\right)\chi_{B_{2R}(0)\setminus B_R(0)} \cdot \rho$$

(Exercise)
where we have used that $|\nabla\rho| \leq \rho\,\frac{|x|}{2(t_0-t)}$.

Inserting $\varphi\cdot\rho$ above, we get

$$\int \varphi\rho\, d\mu_{t_2} + \int_{t_1}^{t_2}\int \varphi\left|\vec{H} + \frac{x^\perp}{2(t_0-t)} - \frac{\nabla^\perp\varphi}{\varphi}\right|^2 \rho\, d\mu_t\, dt$$
$$\leq \int \varphi\rho\, d\mu_{t_1} + \left(\frac{C}{R^2} + \frac{C}{t_0-t_2}\right)\int_{t_1}^{t_2}\int_{B_{2R}(0)\setminus B_R(0)} \rho\, d\mu_t\, dt$$

Note by (**)

$$\sup_{t_1\leq t\leq t_2} \int \rho\, d\mu_t < \infty$$

Then the result follows by the monotone & dominated convergence theorem $\square$

<u>Entropy</u>

For $M^n \subset \mathbb{R}^{n+1}$ define

$$F(M) = \frac{1}{(4\pi)^{n/2}} \int_M e^{-\frac{|x|^2}{4}}\, d\mathcal{H}^n$$

or more generally for a Radon measure μ, we set

$$F_n(\mu) = \frac{1}{(4\pi)^{n/2}} \int e^{-\frac{|x|^2}{4}} d\mu$$

Let $A(r) = \mu(B(0,r))$. Note $A(r)$ is increasing in r and thus $A'(r)$ exists as a Radon measure on $\mathbb{R}^+$.

Then we have by integration by parts (assuming $A(r)$ grows sub-exponentially) that

$$F_n(\mu) = \frac{1}{(4\pi)^{n/2}} \int_0^\infty e^{-\frac{r^4}{4}} A'(r)\,dr = \frac{1}{(4\pi)^{n/2}} \int_0^\infty \frac{\omega_n}{2} r^{n+1} e^{-\frac{r^2}{4}} \underline{\left(\frac{A(r)}{\omega_n r^n}\right)} dr$$

We can also estimate for any $r_0 > 0$

$$F_n(\mu) \geq \frac{1}{(4\pi)^{n/2}} \int_{r_0}^\infty \frac{r}{2} e^{-\frac{r^2}{4}} A(r)\,dr = -\frac{1}{(4\pi)^{\frac{n}{2}}} \int_{r_0}^\infty A(r) \frac{d}{dr}\left(e^{-\frac{r^2}{4}}\right) dr$$

$$\geq \frac{1}{(4\pi)^{n/2}} A(r_0) e^{-\frac{r_0^2}{4}}$$

$\Rightarrow$

$$\frac{\omega_n r^n}{(4\pi)^{n/2}} e^{-\frac{r^2}{4}} \left(\frac{A(r)}{\omega_n r^n}\right) \leq F_n(\mu) \leq C \sup_{r \geq 1} \frac{A(r)}{\omega_n r^n}$$

Colding–Minicozzi defined a related quantity, entropy, as

$$\lambda(M) := \sup_{\substack{\lambda > 0 \\ p \in \mathbb{R}^{n+1}}} F(\lambda M + p)$$

where we define $\lambda_n(\mu)$ correspondingly.

By the above bounds, we see that there exists $C = C(\lambda) > 0$ s.t. for $A(p, r) = \mu(B(p, r))$

$$C^{-1} \sup_{\substack{p \in \mathbb{R}^{n+1} \\ r > 0}} \frac{A(p, r)}{\omega_n r^n} \leq \lambda_u(\mu) \leq C \sup_{\substack{p \in \mathbb{R}^{n+1} \\ r > 0}} \frac{A(p, r)}{\omega_n r^n}$$

Cor: Let $(\mu_t)_{t \geq 0}$ be an integral Brakke flow on $\mathbb{R}^{n+1}$ satisfying $(*)$. Then the entropy $\lambda_u(\mu_t)$ is finite and decreasing w.r.t. t. Furthermore, the area ratios are uniformly controlled for all time.

Tangent flows: Let $(\mu_t)_{t \in I}$ integral Brakke flow. Denote the parabolically rescaled measure around $X = (x_0, t_0)$

$$\mu_t^{X_0, \lambda}(A) = \lambda^n \mu_{t_0 + \lambda^{-2} t}(\lambda^{-1} A + x_0)$$

Check: $\lambda^2 (I - t_0) \ni t \longmapsto \mu_t^{X_0, \lambda}$

again an n-dim integral Brakke flow on $\mathbb{R}^{n+1}$

Note: if the initial flow has entropy bdd by C, then so does $(\mu_t^{X_0, \lambda})$.

Prop (Existence of tangent flows)
For any sequence $\lambda_i \to \infty$, there exists a subsequence (labelled the same) and a Brakke flow $(\nu_t)_{t \in \mathbb{R}}$ s.t.

$$(\mu_t^{X_0, \lambda_i}) \longrightarrow (\nu_t)_{t \in \mathbb{R}}$$

and $$\nu_t(A) = \nu_t^\lambda(A) = \lambda^n \nu_{\lambda^{-2}t}(\lambda^{-1}A) \quad t<0 \quad \forall \lambda>0$$

and ν_{-1} satisfies $$\vec{H} + \frac{x^\perp}{2} = 0 \qquad \nu_{-1}\text{-a.e. } x$$

Furthermore
$$\int \rho_{(0,0)}(\cdot,t)\,\nu_t = \lim_{t'\nearrow t_0} \int \rho_{X_0}\, d\mu_{t'} \qquad t<0$$

<u>Pf</u>: Write $\mu_t^\lambda = \mu_t^{X_0,\lambda}$, $\rho_{(0,0)} = \rho$.

The flows (μ_t^λ) have bounded area ratios by the previous discussion. By compactness, there exists a subsequence (labelled the same) $\lambda_i \to 0$ s.t. $(\mu_t^{\lambda_i}) \to (\nu_t)$ $\forall t \in \mathbb{R}$. Since the flows have uniformly bounded area ratios, for every $t<0$ and $\varepsilon>0$ there exist $R>0$ s.t.

$$\sup_i \int_{\mathbb{R}^{n+1}\setminus B_R(0)} \rho\, d\mu_t^{\lambda_i} < \varepsilon$$

Using a suitable cut-off fct, weak convergence $\mu_t^{\lambda_i} \to \nu_t$ we have

$$\int \rho(\cdot,t)\,d\nu_t = \lim_{i\to\infty} \int \rho(\cdot,t)\,d\mu_t^{\lambda_i}$$

$$\overset{(Ex)}{=} \lim_{t'\to t_0} \int \rho_{X_0}\, d\mu_{t'} \qquad t<0$$

where the last equality follows from the monotonicity formula.

Again by the monotonicity formula for $(\nu_t)_{t\in\mathbb{R}}$ centered at $(0,0)$ yields that for a.e. $t<0$, ν_t is an n-dim. integral varifold with $|\vec{H}| \in \mathcal{L}^2_{loc}(\mu_t)$ and

$$\vec{H} + \frac{x^\perp}{2t} = 0 \qquad \nu_t\text{-a.e. } x.$$

To show self-similarity, define

$$\tilde{\nu}_t(A) = (-t)^{-n/2}\, \nu_t\big((-t)^{-1/2} A\big), \quad t<0$$

It suffices to show that $\tilde{\nu}_t$ is constant in time.

$\phi \in C^2_c(\mathbb{R}^{n+1}; \mathbb{R}^+)$ and $\tilde{\phi}(x,t) = (-t)^{n/2}\phi\big((-t)^{1/2}x\big)$

Note
$$\frac{\partial \tilde{\phi}}{\partial t} = -\frac{n}{2t}\tilde{\phi} - \frac{1}{2t}\langle D\tilde{\phi}, x\rangle$$

By the definition of Brakke flow

$$\int \phi \, d\tilde{\nu}_{t_2} - \int \phi\, d\tilde{\nu}_{t_1} = \int \tilde{\phi}\, d\nu_{t_2} - \int \tilde{\phi}\, d\nu_{t_1}$$

$$\leq \int_{t_1}^{t_2}\int -\frac{n}{2t}\tilde{\phi} - \tilde{\phi}|\vec{H}|^2 + \langle D\tilde{\phi}, \vec{H}\rangle - \frac{1}{2t}\langle D\tilde{\phi}, x\rangle \, d\nu_t\, dt$$

$$= \int_{t_1}^{t_2}\int -\frac{n}{2t}\tilde{\phi} - \frac{\tilde{\phi}}{2t}\langle \vec{H}, x\rangle + \langle D\tilde{\phi}, \frac{x^\perp}{2t}\rangle - \frac{1}{2t}\langle D\tilde{\phi}, x\rangle \, d\nu_t\, dt$$

$$= \int_{t_1}^{t_2}\int -\frac{n}{2t}\tilde{\phi} - \frac{\tilde{\phi}}{2t}\langle \vec{H}, x\rangle - \langle D\tilde{\phi}, \frac{x^T}{2t}\rangle \, d\nu_t\, dt$$

Note $\quad \int -\frac{\hat\phi}{2t}\langle \vec H, x\rangle = \int \frac{1}{2t}\,\mathrm{div}_{\Gamma_t}(\hat\phi x)\, d\nu_t$

$$= \int \frac{n}{2t}\hat\phi + \langle \nabla\hat\phi, \frac{x^T}{2t}\rangle\, d\nu_t$$

$\Longrightarrow \quad \int \phi\, d\hat\nu_t$ non-increasing in t.

Assume w.l.o.g. $\phi < \rho$ and apply the same calculation to $\rho - \phi$ (using exponential decay of ρ + bounded area ratios to validate the insertion of $\rho-\phi$) to see that

$$\int (\rho - \phi)\, d\hat\nu_t \quad \text{is also non-increasing in } t$$

Since $\int \rho\, d\hat\nu_t$ is constant in t

$\Longrightarrow \quad \int \phi\, d\hat\nu_t$ is constant in t. $\square$.