Metric measure spaces satisfying Ricci curvature lower bounds
Lecture 1

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Summer School 2021
Curvature Constraints and Spaces of Metrics

June 14–July 2, 2021
Institut Fourier-Grenoble (hybrid)

European Research Council
Established by the European Commission

Plan of the journey

- Curvature and convergence of Smooth Riemannian manifolds.
- Ricci Curvature for non-smooth spaces: the $CD(K,N), CD^*(K,N), RCD(K,N), RCD^*(K,N)$ conditions.
- Analytic and geometric properties of CD and RCD spaces.
- Applications to smooth Riemannian manifolds.

Gaussian curvature of 2-dimensional surfaces

- Let (Σ, g) be a 2-dimensional Riemannian surface. Denote by ∇ the Levi-Civita connection on (Σ, g).
- Fix $p \in \Sigma$, and $\vec{e}_x = \frac{\partial}{\partial x}, \vec{e}_y = \frac{\partial}{\partial y}$ coordinate basis of $T_p\Sigma$.
- The Gaussian Curvature $K^G_{(\Sigma,g)}(p)$ of (Σ, g) at p is defined by

$$K^G_\Sigma(p) := \frac{g_p\Big((\nabla_{\vec{e}_y}\nabla_{\vec{e}_x} - \nabla_{\vec{e}_x}\nabla_{\vec{e}_y})\vec{e}_x, \vec{e}_y\Big)}{\det(g_p)}$$

- If $(\Sigma, g) \subset \mathbb{R}^3$ is isometrically immersed, then

$$\begin{aligned} K^G_{(\Sigma,g)}(p) &= \text{product of the principal curvatures at } p \\ &= \text{Jacobian of the Gauss map at } p. \end{aligned}$$

- Examples:
  - $0 \equiv$ Gaussian curvature of the Euclidean plane $\mathbb{R}^2$.
  - $\frac{1}{r^2} \equiv$ Gaussian curvature of a 2-dimensional round sphere in $\mathbb{R}^3$ of radius r.
  - $-1 \equiv$ Gaussian curvature of the Hyperbolic plane:
    half plane $\{(x, y) : y > 0\}$ with metric $ds^2 = \frac{dx^2+dy^2}{y^2}$

Sectional and Ricci Curvature Riemannian manifolds

- Let (M^n, g) be an n-dimensional Riemannian manifold.
- Fix $p \in M$ and $\mathrm{span}(\vec{e}_1, \vec{e}_2) = \Pi \subset T_pM$ a 2-dim subspace.
- Let $\Sigma_\Pi = \mathrm{Exp}_p(\Pi \cap B_\varepsilon(0))$=surface obtained by considering all the geodesics starting at p tangent to Π up to length ε. For $\varepsilon > 0$ small enough $\Sigma_\Pi \subset M$ is a smooth 2-dim surface.
- Define the Sectional Curvature of (M, g) at the 2-dim plane $\mathrm{span}(\vec{e}_1, \vec{e}_2) = \Pi \subset T_pM$ as

$$\mathrm{Sec}_p(\vec{e}_1, \vec{e}_2) = K^G_{\Sigma_\Pi}(p) = \text{Gaussian curvature of } \Sigma_\Pi \text{ at } p.$$

- Define the Ricci Curvature of (M, g) at the vector $\vec{v} \in T_pM$ as

$$\mathrm{Ric}_p(\vec{v}, \vec{v}) = |\vec{v}|^2 \sum_{i=1}^{n-1} \mathrm{Sec}_p(\vec{v}, \vec{e}_i) \text{“} = \text{trace of the curvature}''$$

where $\{\vec{e}_1, \dots, \vec{e}_{n-1}, \vec{v}/|\vec{v}|\}$ is an orthonormal basis of (T_pM, g_p).

Some notational remarks on the curvature bounds

Let (M^n, g) be an n-dimensional Riemannian manifold. Denote Sec the sectional curvature and Ric the Ricci curvature.

- For $K \in \mathbb{R}$, we write $\mathrm{Sec} \geq K$ (resp. $\leq K$) if for every $p \in M$ and every 2-dim plane $\Pi \subset T_pM$ it holds $\mathrm{Sec}_p(\Pi) \geq K$ (resp. $\leq K$).
- $\mathrm{Ric}_p : T_pM \times T_pM \to \mathbb{R}$ is a quadratic form. We write $\mathrm{Ric} \geq K$ (resp. $\leq K$) if the quadratic form $\mathrm{Ric}_p - Kg_p$ is non-negative (resp. non-positive) definite at every $p \in M$.
- Examples (model spaces):
  - n-dimensional Euclidean space: $\mathrm{Sec} \equiv 0$, $\mathrm{Ric} \equiv 0$.
  - n-dimensional round sphere of radius 1: $\mathrm{Sec} \equiv 1$, $\mathrm{Ric} \equiv n - 1$.
  - n-dimensional Hyperbolic space: $\mathrm{Sec} \equiv -1$, $\mathrm{Ric} \equiv -(n - 1)$.

Some basics of comparison geometry

Question: (M, g) smooth Riemannian manifold.
If we assume some upper/lower bounds on the sectional or on the Ricci curvature what can we say on the analysis/geometry of (M, g)?

- Upper/Lower bounds on the sectional curvature are strong assumptions with strong implications E.g. Cartan-Hadamard Theorem (if $\mathrm{Sec} \leq 0$ then the universal cover of M is diffeomorphic to $\mathbb{R}^N$), Topogonov triangle comparison theorem($\rightsquigarrow$ definition of Alexandrov spaces: non smooth spaces with upper/lower bounds on the "sectional curvature"), etc.
- Upper bounds on the Ricci curvature are very (too) weak assumptions for geometric conclusions. E.g. Lokhamp Theorem (Gao-Yau, Brooks in dim 3): any closed manifold of dim$\geq$ 3 carries a metric with negative Ricci curvature.

Some basics of comparison geometry: lower Ricci bounds

Lower bounds on the Ricci curvature: a natural framework for comparison geometry (from below)

- Bishop-Gromov volume comparison: (not most general form)
  If (M^n, g) has $\mathrm{Ric} \geq 0$ then for all $x \in M$

  $$R \mapsto \frac{\mathrm{vol_g}(B_R(x))}{\omega_n R^n} \text{ is monotone non-increasing}$$

  In particular, since $\to 1$ as $R \to 0 \rightsquigarrow \mathrm{vol_g}(B_R(x)) \leq \omega_n R^n$.
- Laplacian comparison,
- Cheeger-Gromoll splitting,
- Li-Yau inequalities on heat flow,
- Lévy-Gromov isoperimetric inequality,
- ...

Non smooth setting: Origins of the topic

Gromov in the '80ies

- introduced a notion of convergence for Riemannian manifolds, known as Gromov-Hausdorff convergence (for non-compact manifolds, more convenient a pointed version, called pointed Gromov-Hausdorff convergence $\rightsquigarrow$ GH-convergence of metric balls of every fixed radius)
- observed that a sequence of Riemannian n-dimensional manifolds satisfying a uniform Ricci curvature lower bound is precompact, i.e. it converges up to subsequences to a possibly non-smooth limit space (called, from now on, Ricci limit space)

• Natural question: what can we say about the compactification of the space of Riemannian manifolds with Ricci curvature bounded below (by, say, minus one)?

•Hope: useful also to establish properties for smooth manifolds.

Semi-smooth setting

Ricci limits can have singularities (e.g. a cone, the boundary of a convex body in $\mathbb{R}^n$) which can be dense.

- Cheeger-Colding 1997-2000 three fundamental papers on JDG on the structure of Ricci limit spaces.
  - Collapsing: $\lim_k vol_{g_k}(B_1(\bar{x_k})) = 0 \rightsquigarrow$ loss of dimension in the limit. More difficult, nevertheless they proved that the limit space has a uniquely defined volume measure (up to scaling) and a.e. point has a Euclidean tangent space (a priori, the dimension may vary from point to point). Such points are called regular points, the complementary is called singular set.
  - Non collapsing: $\liminf_k vol_{g_k}(B_1(\bar{x_k})) > 0$. More results: the Hausdorff dimension passes to the limit, one can prove finer estimates on the singular set, e.g. Haudorff codimension 2.
- Colding-Naber, Annals of Math. 2012: the dimension of the tangent space does not change on the regular set, even in the collapsed case.
- Cheeger-Jiang-Naber, Annals of Math. 2021: rectifiability of the singular set in the non-collapsed case.

Extrinsic Vs Intrinsic

- The aforementioned approach to Ricci curvature for non-smooth spaces is a non-intrinsic point of view: consider the non smooth spaces arising as limits of smooth objects. Dichotomy collapsing-non collapsing. Very powerful for structural properties.
- Analogy: like defining $W^{1,2}$ as completion of C^∞ endowed with $W^{1,2}$-norm.
- But $W^{1,2}$ can be defined also in completely intrinsic way without passing via approximations (very convenient for doing calculus of variations).
- GOAL: define in an intrisic-axiomatic way a non smooth space with Ricci curvature bounded below by K and dimension bounded above by N (containing Ricci limits no matter if collapsed or not).
  $\rightsquigarrow$ weak version of a Riemannian manifold with Ric$\geq K$; analogy with GMT (currents, varifolds,etc.)

Preliminary Observation

- sectional curvature bounds for non smooth spaces make perfect sense in metric spaces (X, d) (Alexandrov spaces): sectional curvature is a property of lengths (comparison triangles)
- Ricci curvature is a property of lenghts and volumes: needs also a reference volume measure
  $\rightsquigarrow$ natural setting metric measure spaces $(X, \mathrm{d}, \mathfrak{m})$.

Non smooth setting 1: the Kantorovich-Wasserstein space

Notations:

- $(X, \mathsf{d}, \mathfrak{m})$ complete separable metric space with a σ-finite non-negative Borel measure $\mathfrak{m}$ (more precisely $\mathfrak{m}(B_r(x)) \le c e^{Ar^2}$ for some $A, c > 0$); if we fix a point $\bar{x} \in X$, $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ denotes the corresponding pointed space.
- Let

$$\mathcal{P}_2(X) := \left\{ \mu \,:\, \mu \ge 0,\ \mu(X) = 1,\ \int_X \mathsf{d}^2(x, \bar{x})\, \mu(dx) < \infty \right\}$$

  =Probability measures with finite second moment.
- Given $\mu_1, \mu_2 \in \mathcal{P}_2(X)$, define the (Kantorovich-Wasserstein) quadratic transportation distance

$$W_2^2(\mu_1, \mu_2) := \inf \left\{ \int_{X \times X} \mathsf{d}^2(x, y)\, \gamma(dxdy) \right\}$$

  where $\gamma \in \mathcal{P}(X \times X)$ with $(\pi_i)_\sharp \gamma = \mu_i$, $i = 1, 2$
- $(\mathcal{P}_2(X), W_2)$ is a metric space, geodesic if (X, d) is geodesic

Non smooth setting 2: Entropy functionals

On the metric space $(\mathcal{P}_2(X), W_2)$ consider the following Entropy functionals.

- For $N \in (1, \infty]$, let $\mathcal{U}_{N,\mathfrak{m}}(\mu)$ defined as follows: if $\mu = \rho\mathfrak{m} \ll \mathfrak{m}$

$$\mathcal{U}_{N,\mathfrak{m}}(\rho\mathfrak{m}) := -N \int \rho^{1-\frac{1}{N}} d\mathfrak{m} \quad \text{if } 1 < N < \infty \quad \text{Rényi Entropy}$$

$$\mathcal{U}_{\infty,\mathfrak{m}}(\rho\mathfrak{m}) := \int \rho \log \rho \, d\mathfrak{m} \quad \text{Bolzmann-Shannon Entropy}$$

- if μ is not a.c. then if $N < \infty$ the non a.c. part does not contribute, if $N = +\infty$ then set $\mathcal{U}_{\infty,\mathfrak{m}}(\mu) = \infty$.
- Such entropy functionals show up (and actually were introduced) in statistical mechanics, thermodynamics, information theory, etc.

Non smooth setting: intrinsic-axiomatic definition. 2

- Crucial observation
  [CorderoErausquin-McCann-Schmuckenshlager '01, Otto-Villani '00, Sturm-Von Renesse '05]
  If $(X, \mathsf{d}, \mathfrak{m})$ is a smooth Riemannian manifold (M, g), then $\mathrm{Ric} \geq 0$ (resp. $\geq K$) iff the entropy functional $\mathcal{U}_{\infty,\mathfrak{m}}$ is (K-)convex along geodesics in $(\mathcal{P}_2(X), W_2)$. i.e. for every $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ there exists a W_2-geodesic $(\mu_t)_{t\in[0,1]}$ such that for every $t \in [0, 1]$ it holds

$$\mathcal{U}_{\infty,\mathfrak{m}}(\mu_t) \leq (1-t)\mathcal{U}_{\infty,\mathfrak{m}}(\mu_0)+t\mathcal{U}_{\infty,\mathfrak{m}}(\mu_1)-\frac{K}{2}t(1-t)W_2(\mu_0, \mu_1)^2.$$

- Notice that the notion of (K-)convexity of the Entropy makes sense in a general metric measure space $(X, \mathsf{d}, \mathfrak{m})$.
- DEF of $CD(K, N)$ condition [Lott-Sturm-Villani '06]: fixed $N \in [1, +\infty]$ and $K \in \mathbb{R}$, $(X, \mathsf{d}, \mathfrak{m})$ is a $CD(K, N)$-space if the Entropy $\mathcal{U}_{N,\mathfrak{m}}$ is K-convex along geodesics in $(\mathcal{P}_2(X), W_2)$ (for finite N is a "distorted" (K, N)-geod. conv.).

Non smooth setting: intrinsic-axiomatic definition. 3

Keep in mind:
- $CD(K, N) \rightsquigarrow$ definition Ricci curvature $\geq K$ and dimension $\leq N$ in an intrinsic/axiomatic way for metric measure spaces
- the more convex is $\mathcal{U}_{N,\mathfrak{m}}$ along geodesics in $(\mathcal{P}_2(X), W_2)$, the more the space is positively Ricci curved.

Good properties:

- CONSISTENT: (M, g) satisfies $CD(K, N)$ iff $Ric \geq K$ and $dim(M) \leq N$
- GEOMETRIC PROPERTIES: Brunn-Minkoswki inequality, Bishop-Gromov volume growth, Bonnet-Myers diameter bound, Lichnerowictz spectral gap, etc.
- STABLE under convergence of metric measure spaces? RK: stability will imply that Ricci limits are CD spaces.

Stability of $CD(K,N)$, 1: Lott-Villani Vs Sturm

- Framework of proper spaces (i.e. bounded closed sets are compact), Lott-Villani: $CD(K,N)$ is stable under pointed measured Gromov-Hausdorff convergence (i.e. for every $R > 0$ there is measured Gromov-Hausdorff convergence of balls of radius R around the given points of the space)
- Framework of probability spaces with finite variance (i.e. $\mathfrak{m} \in \mathcal{P}_2(X)$): Sturm defined a distance (also known as Gromov-Wasserstein in the literature)

$$\mathbb{D}\left((X_1, \mathrm{d}_1, \mathfrak{m}_1), (X_2, \mathrm{d}_2, \mathfrak{m}_2)\right) := \inf W_2\left((\iota_1)_\sharp \mathfrak{m}_1, (\iota_2)_\sharp \mathfrak{m}_2\right),$$

inf among all metric spaces (Z, d_Z) and all isometric embeddings $\iota_i(\mathrm{supp}(\mathfrak{m}_i), \mathrm{d}_i) \to (Z, \mathrm{d}_Z)$, $i = 1, 2$. He then proved that $CD(K,N)$ is stable w.r.t. $\mathbb{D}$-convergence.

Stability of $CD(K,N)$, 2: not satisfactory for $N=\infty$

- $CD(K,N)$, for $N<\infty$ implies properness of X, so Lott-Villani fully covers the situation.
- $CD(K,\infty)$ does not imply any sort of compactness, not even local, so pmGH-convergence is quite unnatural. At least for normalized spaces with finite variance Sturm's approach covers the situation.
- In some geometric situations, the assumption that $\mathfrak{m}\in\mathcal{P}_2(X)$ is a bit too restrictive: e.g. when studying blow ups (i.e. tangent cone at a point $\rightsquigarrow$ Cheeger,Colding,Naber) and blow downs (i.e. tangent cones at infinity $\rightsquigarrow$ Cheeger, Colding, Minicozzi, Tian, etc.)
- One may also like to consider sequences of non compact manifolds with diverging dimensions or more generally sequences of spaces with diverging doubling constants.

Q:1) What is a natural notion of convergence in these situations?
2) Is $CD(K,\infty)$ stable w.r.t. this notion?

Pointed measured Gromov (pmG for short) convergence

DEF:(Gigli-M.-Savaré) $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n) \to (X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ in pmG-sense if there exist a complete and separable space (Z, d_z) and isometric embeddings $\iota_n : X_n \to Z$, $n \in \bar{N} := \mathbb{N} \cup \{\infty\}$ s.t.

$$\int \varphi \, (\iota_n)_\sharp \mathfrak{m}_n \to \int \varphi \, (\iota_\infty)_\sharp \mathfrak{m}_\infty, \ \forall \varphi \in C_{bs}(Z), \text{ where}$$

$C_{bs}(Z) := \{f : Z \to \mathbb{R}$ cont., bounded with bounded support $\}$.

- The definition above is extrinsic but we prove it can be characterized in a (maybe less immediate) totally intrinsic way, according various equivalent approaches (via a pointed version of Gromov reconstruction Theorem or via a pointed/weighted version of Sturm's $\mathbb{D}$-distance).
- On doubling spaces pmG-convergence above is equivalent to mGH-convergence ($\rightsquigarrow$ consistent with Lott-Villani).
- On normalized spaces of finite variance pmG-convergence is equivalent to $\mathbb{D}$-convergence ($\rightsquigarrow$ consistent with Sturm).
- pmG-convergence no a priori assumption on $(X_n, \mathsf{d}_n, \mathfrak{m}_n)$.

$CD(K,\infty)$ is stable under pmG-convergence

THM(Gigli-M.-Savaré): Let $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$, $n \in \mathbb{N}$, be a sequence of $CD(K,\infty)$ p.m.m. spaces converging to $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ in the pmG-sense. Then $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty)$ is a $CD(K,\infty)$ space as well.

Rough sketch of Proof (Borrowed from Lott-Sturm-Villani):

1. Prove Γ-convergence of $\mathcal{U}_{\infty,\mathfrak{m}}$ under pmG-convergence:
   - "$\Gamma - \liminf$ inequality": For every $\mu^\infty \in \mathcal{P}_2(X_\infty)$ and every $\mu^n \in \mathcal{P}_2(X_n)$ such that $\mu^n \to \mu^\infty$ weakly in Z, it holds that
   $$\mathcal{U}_{\infty,\mathfrak{m}}(\mu^\infty|\mathfrak{m}_\infty) \leq \liminf_{n\to\infty} \mathcal{U}_{\infty,\mathfrak{m}}(\mu^n|\mathfrak{m}_n).$$
   - "Existence of a recovery sequence": For every $\mu^\infty \in \mathcal{P}_2(X_\infty)$ there exist $\mu^n \in \mathcal{P}_2(X_n)$ such that $\mu^n \to \mu^\infty$ weakly in Z and
   $$\mathcal{U}_{\infty,\mathfrak{m}}(\mu^\infty|\mathfrak{m}_\infty) \geq \limsup_{n\to\infty} \mathcal{U}_{\infty,\mathfrak{m}}(\mu^n|\mathfrak{m}_n).$$
2. Use the compactness of $\mathfrak{m}_n$ to prove compactness of Wasserstein-geodesics in the converging spaces
3. Conclude that K-geodesic convexity is preserved.