Metric measure spaces satisfying Ricci curvature lower bounds
Lecture 1

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Plan of the journey

- Curvature and convergence of Smooth Riemannian manifolds.
- Analytic and geometric properties of $\text{CD}$ and $\text{RCD}$ spaces.
- Applications to smooth Riemannian manifolds.
Gaussian curvature of 2-dimensional surfaces

- Let \((\Sigma, g)\) be a 2-dimensional Riemannian surface. Denote by \(\nabla\) the Levi-Civita connection on \((\Sigma, g)\).
- Fix \(p \in \Sigma\), and \(\vec{e}_x = \frac{\partial}{\partial x}, \vec{e}_y = \frac{\partial}{\partial y}\) coordinate basis of \(T_p \Sigma\).
- The Gaussian Curvature \(K^G_{(\Sigma,g)}(p)\) of \((\Sigma, g)\) at \(p\) is defined by
  \[
  K^G_{\Sigma}(p) := \frac{g_p \left( (\nabla_{\vec{e}_y} \nabla_{\vec{e}_x} - \nabla_{\vec{e}_x} \nabla_{\vec{e}_y}) \vec{e}_x, \vec{e}_y \right)}{\det(g_p)}
  \]
- If \((\Sigma, g) \subset \mathbb{R}^3\) is isometrically immersed, then
  \(K^G_{(\Sigma,g)}(p)\) = product of the principal curvatures at \(p\)
  = Jacobian of the Gauss map at \(p\).
- Examples:
  - \(0 \equiv\) Gaussian curvature of the Euclidean plane \(\mathbb{R}^2\).
  - \(\frac{1}{r^2} \equiv\) Gaussian curvature of a 2-dimensional round sphere in \(\mathbb{R}^3\) of radius \(r\).
  - \(-1 \equiv\) Gaussian curvature of the Hyperbolic plane:
    half plane \(\{(x, y) : y > 0\}\) with metric \(ds^2 = \frac{dx^2 + dy^2}{y^2}\)
Let \((M^n, g)\) be an \(n\)-dimensional Riemannian manifold.

Fix \(p \in M\) and \(\text{span} (\vec{e}_1, \vec{e}_2) = \Pi \subset T_p M\) a 2-dim subspace.

Let \(\Sigma_\Pi = \text{Exp}_p (\Pi \cap B_\varepsilon (0))\) = surface obtained by considering all the geodesics starting at \(p\) tangent to \(\Pi\) up to length \(\varepsilon\). For \(\varepsilon > 0\) small enough \(\Sigma_\Pi \subset M\) is a smooth 2-dim surface.

Define the **Sectional Curvature** of \((M, g)\) at the 2-dim plane \(\text{span} (\vec{e}_1, \vec{e}_2) = \Pi \subset T_p M\) as

\[
\text{Sec}_p (\vec{e}_1, \vec{e}_2) = K^G_{\Sigma_\Pi} (p) = \text{Gaussian curvature of } \Sigma_\Pi \text{ at } p.
\]

Define the **Ricci Curvature** of \((M, g)\) at the vector \(\vec{v} \in T_p M\) as

\[
\text{Ric}_p (\vec{v}, \vec{v}) = |\vec{v}|^2 \sum_{i=1}^{n-1} \text{Sec}_p (\vec{v}, \vec{e}_i) = \text{trace of the curvature}
\]

where \(\{\vec{e}_1, \ldots, \vec{e}_{n-1}, \vec{v}/|\vec{v}|\}\) is an orthonormal basis of \((T_p M, g_p)\).
Some notational remarks on the curvature bounds

Let \((M^n, g)\) be an \(n\)-dimensional Riemannian manifold. Denote \(\text{Sec}\) the sectional curvature and \(\text{Ric}\) the Ricci curvature.

- For \(K \in \mathbb{R}\), we write \(\text{Sec} \geq K\) (resp. \(\leq K\)) if for every \(p \in M\) and every 2-dim plane \(\Pi \subset T_pM\) it holds \(\text{Sec}_p(\Pi) \geq K\) (resp. \(\leq K\)).

- \(\text{Ric}_p : T_pM \times T_pM \to \mathbb{R}\) is a quadratic form. We write \(\text{Ric} \geq K\) (resp. \(\leq K\)) if the quadratic form \(\text{Ric}_p - Kg_p\) is non-negative (resp. non-positive) definite at every \(p \in M\).

- **Examples (model spaces):**
  - \(n\)-dimensional Euclidean space: \(\text{Sec} \equiv 0, \text{Ric} \equiv 0\).
  - \(n\)-dimensional round sphere of radius 1: \(\text{Sec} \equiv 1, \text{Ric} \equiv n - 1\).
  - \(n\)-dimensional Hyperbolic space: \(\text{Sec} \equiv -1, \text{Ric} \equiv -(n - 1)\).
Some basics of comparison geometry

**Question:** \((M, g)\) smooth Riemannian manifold.
If we assume some upper/lower bounds on the sectional or on the
Ricci curvature what can we say on the analysis/geometry of
\((M, g)\)?

- **Upper/Lower bounds on the sectional curvature** are strong
assumptions with strong implications. E.g. Cartan-Hadamard
Theorem (if \(\text{Sec} \leq 0\) then the universal cover of \(M\) is
diffeomorphic to \(\mathbb{R}^N\)), Topogonov triangle comparison
theorem\((\leadsto\) definition of Alexandrov spaces: non smooth
spaces with upper/lower bounds on the ”sectional
curvature”), etc.

- **Upper bounds on the Ricci curvature** are very (too) weak
assumptions for geometric conclusions. E.g. Lokhamp
Theorem (Gao-Yau, Brooks in dim 3): any closed manifold of
dim \(\geq 3\) carries a metric with negative Ricci curvature.
Some basics of comparison geometry: lower Ricci bounds

Lower bounds on the Ricci curvature: a natural framework for comparison geometry (from below)

- Bishop-Gromov volume comparison: (not most general form)
  If \((M^n, g)\) has \(\text{Ric} \geq 0\) then for all \(x \in M\)

\[
R \mapsto \frac{\text{vol}_g(B_R(x))}{\omega_n R^n}
\]

is monotone non-increasing

In particular, since \(\to 1\) as \(R \to 0\) \(\sim \) \(\text{vol}_g(B_R(x)) \leq \omega_n R^n\).

- Laplacian comparison,
- Cheeger-Gromoll splitting,
- Li-Yau inequalities on heat flow,
- Lévy-Gromov isoperimetric inequality,
- ...
Non smooth setting: Origins of the topic

Gromov in the ’80ies

▶ introduced a notion of convergence for Riemannian manifolds, known as Gromov-Hausdorff convergence (for non-compact manifolds, more convenient a pointed version, called pointed Gromov-Hausdorff convergence \(\rightarrow\) GH-convergence of metric balls of every fixed radius)

▶ observed that a sequence of Riemannian \(n\)-dimensional manifolds satisfying a uniform Ricci curvature lower bound is precompact, i.e. it converges up to subsequences to a possibly non-smooth limit space (called, from now on, Ricci limit space)

● Natural question: what can we say about the compactification of the space of Riemannian manifolds with Ricci curvature bounded below (by, say, minus one)?
● Hope: useful also to establish properties for smooth manifolds.
Ricci limits can have singularities (e.g. a cone, the boundary of a convex body in $\mathbb{R}^n$) which can be dense.

- **Cheeger-Colding** 1997-2000 three fundamental papers on JDG on the structure of Ricci limit spaces.
  - **Collapsing**: $\lim_k \text{vol}_{g_k}(B_1(\bar{x}_k)) = 0 \Rightarrow$ loss of dimension in the limit. More difficult, nevertheless they proved that the limit space has a uniquely defined volume measure (up to scaling) and a.e. point has a Euclidean tangent space (a priori, the dimension may vary from point to point). Such points are called regular points, the complementary is called singular set.
  - **Non collapsing**: $\liminf_k \text{vol}_{g_k}(B_1(\bar{x}_k)) > 0$. More results: the Hausdorff dimension passes to the limit, one can prove finer estimates on the singular set, e.g. Hausdorff codimension 2.

- **Colding-Naber**, Annals of Math. 2012: the dimension of the tangent space does not change on the regular set, even in the collapsed case.

The aforementioned approach to Ricci curvature for non-smooth spaces is a non-intrinsic point of view: consider the non smooth spaces arising as limits of smooth objects. Dichotomy collapsing-non collapsing. Very powerful for structural properties.

Analogy: like defining $W^{1,2}$ as completion of $C^\infty$ endowed with $W^{1,2}$-norm.

But $W^{1,2}$ can be defined also in completely intrinsic way without passing via approximations (very convenient for doing calculus of variations).

GOAL: define in an intrinsic-axiomatic way a non smooth space with Ricci curvature bounded below by $K$ and dimension bounded above by $N$ (containing Ricci limits no matter if collapsed or not).

weak version of a Riemannian manifold with $\text{Ric} \geq K$; analogy with GMT (currents, varifolds, etc.)
sectional curvature bounds for non smooth spaces make perfect sense in metric spaces \((X, d)\) (Alexandrov spaces): sectional curvature is a property of lengths (comparison triangles)

Ricci curvature is a property of lengths and volumes: needs also a reference volume measure

natural setting metric measure spaces \((X, d, m)\).
Non smooth setting 1: the Kantorovich-Wasserstein space

Notations:

▶ $(X, d, m)$ complete separable metric space with a $\sigma$-finite non-negative Borel measure $m$ (more precisely $m(B_r(x)) \leq ce^{Ar^2}$ for some $A, c > 0$); if we fix a point $\bar{x} \in X$, $(X, d, m, \bar{x})$ denotes the corresponding pointed space.

▶ Let $P_2(X) := \left\{ \mu : \mu \geq 0, \mu(X) = 1, \int_X d^2(x, \bar{x}) \mu(dx) < \infty \right\}$

$= \text{Probability measures with finite second moment.}$

▶ Given $\mu_1, \mu_2 \in P_2(X)$, define the (Kantorovich-Wasserstein) quadratic transportation distance

$$W_2^2(\mu_1, \mu_2) := \inf \left\{ \int_{X \times X} d^2(x, y) \gamma(dx dy) \right\}$$

where $\gamma \in \mathcal{P}(X \times X)$ with $(\pi_i)_\# \gamma = \mu_i, i = 1, 2$

▶ $(P_2(X), W_2)$ is a metric space, geodesic if $(X, d)$ is geodesic
On the metric space \((\mathcal{P}_2(X), W_2)\) consider the following Entropy functionals.

- For \(N \in (1, \infty]\), let \(U_{N,m}(\mu)\) defined as follows: if \(\mu = \rho m \ll m\)

\[
U_{N,m}(\rho m) := -N \int \rho^{1 - \frac{1}{N}} \, dm \quad \text{if } 1 < N < \infty \quad \text{Rényi Entropy}
\]

\[
U_{\infty,m}(\rho m) := \int \rho \log \rho \, dm \quad \text{Bolzmann-Shannon Entropy}
\]

- If \(\mu\) is not a.c. then if \(N < \infty\) the non a.c. part does not contribute, if \(N = +\infty\) then set \(U_{\infty,m}(\mu) = \infty\).

- Such entropy functionals show up (and actually were introduced) in statistical mechanics, thermodynamics, information theory, etc.
Crucial observation

[CorderoErausquin-McCann-Schmuckenschlager ’01, Otto-Villani ’00, Sturm-Von Renesse ’05]
If \((X, d, m)\) is a smooth Riemannian manifold \((M, g)\), then \(\text{Ric} \geq 0\) (resp. \(\geq K\)) iff the entropy functional \(\mathcal{U}_{\infty, m}\) is \((K-)\)convex along geodesics in \((\mathcal{P}_2(X), W_2)\). i.e. for every \(\mu_0, \mu_1 \in \mathcal{P}_2(X)\) there exists a \(W_2\)-geodesic \((\mu_t)_{t \in [0, 1]}\) such that for every \(t \in [0, 1]\) it holds

\[
\mathcal{U}_{\infty, m}(\mu_t) \leq (1-t)\mathcal{U}_{\infty, m}(\mu_0) + t\mathcal{U}_{\infty, m}(\mu_1) - \frac{K}{2} t(1-t)W_2(\mu_0, \mu_1)^2.
\]

Notice that the notion of \((K-)\)convexity of the Entropy makes sense in a general metric measure space \((X, d, m)\).

DEF of \(CD(K, N)\) condition [Lott-Sturm-Villani ’06]: fixed \(N \in [1, +\infty]\) and \(K \in \mathbb{R}\), \((X, d, m)\) is a \(CD(K, N)\)-space if the Entropy \(\mathcal{U}_{N, m}\) is \(K\)-convex along geodesics in \((\mathcal{P}_2(X), W_2)\) (for finite \(N\) is a “distorted” \((K, N)\)-geod. conv.).
Keep in mind:
- $CD(K, N) \leftrightarrow$ definition Ricci curvature $\geq K$ and dimension $\leq N$ in an intrinsic/axiomatic way for metric measure spaces
- the more convex is $\mathcal{U}_{N,m}$ along geodesics in $(\mathcal{P}_2(X), W_2)$, the more the space is positively Ricci curved.

Good properties:

- CONSISTENT: $(M, g)$ satisfies $CD(K, N)$ iff $Ric \geq K$ and $\text{dim}(M) \leq N$
- GEOMETRIC PROPERTIES: Brunn-Minkoswki inequality, Bishop-Gromov volume growth, Bonnet-Myers diameter bound, Lichnerowictz spectral gap, etc.
- STABLE under convergence of metric measure spaces? RK: stability will imply that Ricci limits are CD spaces.
Stability of $\text{CD}(K, N)$, 1: Lott-Villani Vs Sturm

- Framework of **proper** spaces (i.e. bounded closed sets are compact), **Lott-Villani**: $\text{CD}(K, N)$ is stable under pointed measured Gromov-Hausdorff convergence (i.e. for every $R > 0$ there is measured Gromov-Hausdorff convergence of balls of radius $R$ around the given points of the space)

- Framework of **probability spaces with finite variance** (i.e. $m \in \mathcal{P}_2(X)$): **Sturm** defined a distance (also known as Gromov-Wasserstein in the literature)

$$\mathbb{D}((X_1, d_1, m_1), (X_2, d_2, m_2)) := \inf W_2 ((\nu_1)_\# m_1, (\nu_2)_\# m_2),$$

inf among all metric spaces $(Z, d_Z)$ and all isometric embeddings $\nu_i(\text{supp}(m_i), d_i) \to (Z, d_Z), i = 1, 2$. He then proved that $\text{CD}(K, N)$ is stable w.r.t. $\mathbb{D}$-convergence.
Stability of $CD(K, N)$, 2: not satisfactory for $N = \infty$

- $CD(K, N)$, for $N < \infty$ implies properness of $X$, so Lott-Villani fully covers the situation.
- $CD(K, \infty)$ does not imply any sort of compactness, not even local, so pmGH-convergence is quite unnatural. At least for normalized spaces with finite variance Sturm’s approach covers the situation.
- In some geometric situations, the assumption that $\mathfrak{m} \in \mathcal{P}_2(X)$ is a bit too restrictive: e.g. when studying blow ups (i.e. tangent cone at a point $\rightsquigarrow$ Cheeger, Colding, Naber) and blow downs (i.e. tangent cones at infinity $\rightsquigarrow$ Cheeger, Colding, Minicozzi, Tian, etc.)
- One may also like to consider sequences of non compact manifolds with diverging dimensions or more generally sequences of spaces with diverging doubling constants.

Q: 1) What is a natural notion of convergence in these situations?

2) Is $CD(K, \infty)$ stable w.r.t. this notion?
Pointed measured Gromov (pmG for short) convergence

**DEF:** (Gigli-M.-Savaré) $(X_n, d_n, m_n, \bar{x}_n) \rightarrow (X_\infty, d_\infty, m_\infty, \bar{x}_\infty)$ in pmG-sense if there exist a complete and separable space $(Z, d_Z)$ and isometric embeddings $\iota_n : X_n \rightarrow Z$, $n \in \tilde{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ s.t.

$$\int \varphi(\iota_n)_\# m_n \rightarrow \int \varphi(\iota_\infty)_\# m_\infty, \ \forall \varphi \in C_{bs}(Z),$$

where $C_{bs}(Z) := \{f : Z \rightarrow \mathbb{R} \text{ cont., bounded with bounded support}\}$.

- The definition above is extrinsic but we prove it can be characterized in a (maybe less immediate) totally intrinsic way, according various equivalent approaches (via a pointed version of Gromov reconstruction Theorem or via a pointed/weighted version of Sturm’s $D$-distance).
- On doubling spaces pmG-convergence above is equivalent to mGH-convergence ($\sim \Rightarrow$ consistent with Lott-Villani).
- On normalized spaces of finite variance pmG-convergence is equivalent to $D$-convergence ($\sim \Rightarrow$ consistent with Sturm).
- pmG-convergence no a priori assumption on $(X_n, d_n, m_n)$. 
\( CD(K, \infty) \) is stable under \( pmG \)-convergence

**THM (Gigli-M.-Savaré):** Let \((X_n, d_n, m_n, \bar{x}_n)\), \( n \in \mathbb{N} \), be a sequence of \( CD(K, \infty) \) p.m.m. spaces converging to \((X_\infty, d_\infty, m_\infty, \bar{x}_\infty)\) in the \( pmG \)-sense. Then \((X_\infty, d_\infty, m_\infty)\) is a \( CD(K, \infty) \) space as well.

**Rough sketch of Proof** (Borrowed from Lott-Sturm-Villani):

1. Prove \( \Gamma \)-convergence of \( \mathcal{U}_{\infty, m} \) under \( pmG \)-convergence:
   - "\( \Gamma - \lim \inf \) inequality": For every \( \mu^\infty \in \mathcal{P}_2(X_\infty) \) and every \( \mu^n \in \mathcal{P}_2(X_n) \) such that \( \mu^n \rightharpoonup \mu^\infty \) weakly in \( Z \), it holds that
     \[
     \mathcal{U}_{\infty, m}(\mu^\infty | m_\infty) \leq \lim_{n \to \infty} \mathcal{U}_{\infty, m}(\mu^n | m_n).
     \]
   - "Existence of a recovery sequence": For every \( \mu^\infty \in \mathcal{P}_2(X_\infty) \) there exist \( \mu^n \in \mathcal{P}_2(X_n) \) such that \( \mu^n \rightharpoonup \mu^\infty \) weakly in \( Z \) and
     \[
     \mathcal{U}_{\infty, m}(\mu^\infty | m_\infty) \geq \lim_{n \to \infty} \sup \mathcal{U}_{\infty, m}(\mu^n | m_n).
     \]

2. Use the compactness of \( m_n \) to prove compactness of Wasserstein-geodesics in the converging spaces

3. Conclude that \( K \)-geodesic convexity is preserved.