Metric measure spaces satisfying Ricci curvature lower bounds

Lecture 2

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Recall from Lecture 1: Def of $CD(K, N)$

- **$CD(K, N)$ condition** [Lott-Sturm-Villani ’06]: fixed $N \in [1, +\infty]$ and $K \in \mathbb{R}$, $(X, d, m)$ is a $CD(K, N)$-space if the Entropy $\mathcal{U}_{N,m}$ is $K$-convex along geodesics in $(\mathcal{P}_2(X), W_2)$ (for finite $N$ is a “distorted” $(K, N)$-geod. conv.).

Cases easier to write, to keep in mind:

- **$CD(K, \infty)$**: for every $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ there exists a $W_2$-geodesic $(\mu_t)_{t \in [0, 1]}$ such that for every $t \in [0, 1]$ it holds

  \[ \mathcal{U}_{\infty,m}(\mu_t) \leq (1-t)\mathcal{U}_{\infty,m}(\mu_0) + t\mathcal{U}_{\infty,m}(\mu_1) - \frac{K}{2} t(1-t)W_2(\mu_0, \mu_1)^2. \]

- **$CD(0, N)$**: for every $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ there exists a $W_2$-geodesic $(\mu_t)_{t \in [0, 1]}$ such that for every $t \in [0, 1]$ it holds

  \[ \mathcal{U}_{N,m}(\mu_t) \leq (1 - t)\mathcal{U}_{N,m}(\mu_0) + t\mathcal{U}_{N,m}(\mu_1). \]
Recall from Lecture 1: stability of $CD(K, N)$

**THM:** Fix $K \in \mathbb{R}$ and $N \in [1, \infty]$. Let $(X_n, d_n, m_n, \bar{x}_n), \ n \in \mathbb{N}$, be a sequence of $CD(K, N)$ p.m.m. spaces converging to $(X_\infty, d_\infty, m_\infty, \bar{x}_\infty)$ in the pmGH-sense (or, more generally, in the pmG sense).
Then $(X_\infty, d_\infty, m_\infty)$ is a $CD(K, N)$ space as well.

From the stability of $CD(K, N)$: any pmGH limit of a sequence of Riemannian manifolds with $\text{Ricci} \geq K$ and $\text{dim} \leq N$ is a $CD(K, N)$ space. $\rightsquigarrow CD(K, N) \supset \text{Ricci limits}$. 
Plan of Lecture 2

- Introduce $RCD(K, \infty)$
- Discuss gradient flows formulation of $RCD(K, \infty)$
- Finite dimensional theory: $RCD(K, N)$ and $RCD^*(K, N)$
- Bochner inequality and some applications
Part 1: $RCD(K, \infty)$ spaces
**CD**(\(K, N\)) Vs Ricci limits

- From the stability of \(CD(K, N)\): any pmGH limit of a sequence of Riemannian manifolds with Ricci \(\geq K\) and \(\leq N\) is a \(CD(K, N)\) space. \(\rightsquigarrow CD(K, N) \supset\) Ricci limits.

- Finsler manifolds with Ricci curvature bounded below are \(CD(K, N)\). E.g. \((\mathbb{R}^N, \| \cdot \|, \mathcal{L}^N)\) is \(CD(0, N)\), \(\| \cdot \|\) any norm.

- FACT: If a smooth Finsler manifold \(M\) is a Ricci-limit space then \(M\) is Riemannian (Cheeger-Colding '00). \(\rightsquigarrow\) the class of \(CD(K, N)\) is, in some sense, too large.

- Moreover, and maybe more importantly, some fundamental theorems in comparison geometry of Riemannian manifolds (e.g. Cheeger-Gromoll Splitting Theorem) are not true in the larger Finsler category (e.g. \((\mathbb{R}^2, \| \cdot \|_{\infty})\) is \(CD(0, 2)\), contains a line but does not split isometrically).

- \(\rightsquigarrow\) We wish to reinforce the \(CD(K, N)\) condition in order to isolate the “Riemannian” \(CD(K, N)\) spaces; in other words, we wish to rule out Finsler structures, but in a sufficiently weak way to still get a STABLE notion under pmGH converg.
Cheeger energy and $RCD(K, \infty)$ spaces

- Given a m.m.s. $(X, d, m)$ and $f \in L^2(X, m)$, define the Cheeger energy

$$Ch_m(f) := \frac{1}{2} \int_X |\nabla f|_w^2 \, dm = \liminf_{u \rightarrow f \text{ in} L^2} \frac{1}{2} \int_X (\text{lip} u)^2 \, dm$$

where $|\nabla f|_w$ is the minimal weak upper gradient.

- **Crucial observation:** On a Finsler manifold $M$, the Cheeger energy is quadratic (i.e. parallelogram identity holds) iff $M$ is Riemannian. A m.m.s. where the Cheeger energy is quadratic is said infinitesimally Hilbertian.

- **Idea** Reinforce the $CD$ condition by asking that the Cheeger energy is quadratic.

**DEF** (Ambrosio-Gigli-Savaré, see also Ambrosio-Gigli-M.-Rajala) Given $K \in \mathbb{R}$, $(X, d, m)$ is an $RCD(K, \infty)$ space if it is a $CD(K, \infty)$ space & infinitesimally Hilbertian.

**Question:** is $RCD(K, \infty)$ stable under pmG-convergence?
Stability of heat flow under pmG-convergence

- $Ch_m : L^2(X, m) \to \mathbb{R}$ is a convex and l.s.c. functional so (by classical theory of gradient flows, e.g. Brezis) admits a unique gradient flow $(H_t)_{t \geq 0}$ called Heat flow.
- If $(X_n, d_n, m_n, \bar{x}_n) \to (X_\infty, d_\infty, m_\infty, \bar{x}_\infty)$ in the pmG-sense, then there is a way to define convergence of a sequence $f_n \in L^2(X_n, m_n)$ to a function $f_\infty \in L^2(X_\infty, m_\infty)$.

**THM (Gigli '11-Gigli-M-Savaré '13)** [Stability of Heat flows]
Let $(X_n, d_n, m_n, \bar{x}_n) \to (X_\infty, d_\infty, m_\infty, \bar{x}_\infty)$ in the pmG-sense, $X_n$ are $CD(K, \infty)$-spaces. If $f_n \in L^2(X_n, m_n)$ strongly $L^2$-converges to $f_\infty \in L^2(X_\infty, m_\infty)$, then

$$H^n_t(f_n) \to H_\infty(f_\infty)$$ strongly in $L^2$ for every $t \geq 0$.

**Idea of proof**: i) Mosco convergence of Cheeger energies under pmG-convergence (pass via the entropy).
ii) convergence of resolvant maps
iii) approximate the heat flow by iterated resolvant maps to conclude.
Fact: $(X, d, m)$ is infinitesimally Hilbertian iff $H_t : L^2(X, m) \to L^2(X, m)$ is linear for every $t > 0$.

**THM** (Ambrosio-Gigli-Savaré '11, Gigli-M-Savaré '13): Let $(X_n, d_n, m_n, \bar{x}_n), n \in \mathbb{N}$, be a sequence of $RCD(K, \infty)$ p.m.m. spaces converging to a limit space $(X_\infty, d_\infty, m_\infty, \bar{x}_\infty)$ in the pmG-sense. Then $(X_\infty, d_\infty, m_\infty)$ is $RCD(K, \infty)$ as well.

**Idea of proof:**

i) we already know that $CD(K, \infty)$ is stable, so $(X_\infty, d_\infty, m_\infty)$ is a $CD(K, \infty)$ space.

ii) since the heat flows of $X_n$ are linear, by the stability of heat flows also the limit heat flow is linear.
Let \((Y, d_{\mathcal{Y}})\) be a geodesic space (later we will take \((Y, d_{\mathcal{Y}}) = (\mathcal{P}_2(X), W_2))\). A functional \(E : Y \to \mathbb{R} \cup \{+\infty\}\) is said **\(K\)-convex** if for every \(y_0, y_1 \in Y\) there exists a constant speed geodesic \(\gamma : [0, 1] \to Y\) such that

\[
\gamma_0 = y_0 \quad \text{and} \quad \gamma_1 = y_1 \quad \text{and} \quad E(\gamma_t) \leq (1 - t)E(y_0) + tE(y_1) - \frac{K}{2} t(1 - t)d^2_{\mathcal{Y}}(y_0, y_1)
\]

**\(f\)** smooth **\(K\)-convex** function on \(\mathbb{R}^n\), then \(v = -\nabla f(x)\) iff

\[
<v, x - y> + \frac{K}{2}|x - y|^2 + f(x) \leq f(y) \quad \forall y \in \mathbb{R}^n
\]

so \((x_t)\) is a gradient flow of \(f\), i.e. \(x_t' = -\nabla f(x)\) iff

\[
\frac{d}{dt} \frac{|x_t - y|^2}{2} + \frac{K}{2}|x_t - y|^2 + f(x_t) \leq f(y) \quad \forall y \in \mathbb{R}^n, t \geq 0.
\]
Definition of EVI\(K\) flow: Let \((Y, d_Y)\) be a geodesic space and \(E\) a l.s.c. functional. A locally absolutely continuous curve \((y_t)\) with \(y_t \in D(E)\) for every \(t > 0\) is said EVI\(K\) gradient flow of \(E\) if

\[
\frac{d}{dt} \left[ \frac{d^2_Y(y_t, z)}{2} + \frac{K}{2} d^2_Y(y_t, z) + E(y_t) \right] \leq E(z) \quad \forall z \in Y, \text{a.e.} t > 0.
\]

Remark: the existence of an EVI\(K\) flow of \(E\) depends on both the \(K\)-convexity of \(E\) and the infinitesimal Hilbertianity of the space; existence is not true in general, but in case of existence then nice contractivity and regularizing properties of the flow hold.
$RCD(K,\infty)$ is equivalent to $EVI_K$

**Theorem**[Ambrosio-Gigli-Savaré(2011)-Ambrosio-Gigli-M.-Rajala(2012)]: $(X,d,m)$ is $RCD(K,\infty)$ iff for every $\mu \in \mathcal{P}_2(X)$ with $\text{supp}(\mu) \subset \text{supp}(m)$ there exists an $EVI_K$ gradient flow $(\mu_t)$ of $\mathcal{U}_{\infty,m}$ in $(\mathcal{P}_2(X), W_2)$ starting from $\mu$.

**Remark**

- $EVI_K$ formulation useful for proving the stability of $RCD(K,\infty)$ [Ambrosio-Gigli-Savaré(2011)] for $m \in \mathcal{P}(X)$ and [Gigli-M.-Savaré(2013)] for general $m$ and without any compactness assumption.

- $EVI_K$ formulation useful to prove that $RCD(K,\infty)$ is equivalent to $BE(K,\infty)$, i.e. Bochner inequality for $N = \infty$ [Ambrosio-Gigli-Savaré(2012)]. (See later for more on Bochner inequality)
Finite dimensional Theory: $RCD^*(K, N)$ and $RCD(K, N)$ spaces
The reduced curvature-dimension condition $CD^*(K, N)$ was introduced by Bacher-Sturm (2010).

Modification of $CD(K, N)$: (a priori) weaker convexity condition on $\mathcal{U}_{N,m}$

$CD(K, N) \Rightarrow CD^*(K, N) \Rightarrow CD(K^*, N)$ where $K^* = \frac{K(N-1)}{N}$

If $(X,d)$ is non branching then (local to global) $CD^*(K, N) \iff CD^*_{loc}(K, N) \iff CD_{loc}(K^-, N)$.

In non branching spaces tensorization holds.

Same geometric consequence of $CD(K, N)$ (Bishop-Gromov, Bonnet-Myers, Lichnerowicz) but sometimes with slightly worse constants.
**RCD\(^\ast\)(K, N) and RCD(K, N) spaces**

**DEF:** \((X, d, m)\) is \(RCD\(^\ast\)(K, N)\) (resp. \(RCD(K, N)\)) iff Infinit. Hilbertian \(CD\(^\ast\)(K, N)\) (resp. \(CD(K, N)\)).

**Stability:** If \((X_i, d_i, m_i, \bar{x}_i)\) are \(RCD\(^\ast\)(K, N)\) (resp. \(RCD(K, N)\)) and converge in pmGH (or pmG) to \((X, d, m, \bar{x})\), then \((X, d, m)\) is \(RCD\(^\ast\)(K, N)\) (resp. \(RCD(K, N)\)) as well.

\(\leadsto\) any pmGH limit of a sequence of Riemannian manifolds with \(\text{Ricci} \geq K\) and \(\text{dim} \leq N\) is a \(RCD(K, N)\) space

\(RCD(K, \infty) \supset RCD\(^\ast\)(K, N) \supset RCD(K, N) \supset \text{Ricci limits of } N\text{-dim man}\)

for every \(N \in \mathbb{N}\).

**Q:** are the inclusions above strict or not?
About

\[ \text{RCD}(K, \infty) \supset \text{RCD}^*(K, N) \supset \text{RCD}(K, N) \supset \text{Ricci limits} \]

- \[ \text{RCD}(K, \infty) \nsubseteq \text{RCD}^*(K, N) \].
  Example: Gaussian space \((\mathbb{R}^n, \| \cdot \|_{\text{eucl}}, \gamma_n)\)
- THM (Cavalletti-Milman, Inventiones Math. 2021): In case \(m(X) < \infty\) it holds \(\text{RCD}^*(K, N) \Leftrightarrow \text{RCD}(K, N)\).
  No reason to expect if fails for \(m(X) = \infty\) but adaptation of proof is not trivial.

- \[ \text{RCD}(K, N) \nsubseteq \text{pmGH limits of Riem. man. with Ricci} \geq K \text{ and dimension} \leq N \].
  Example: \(C(\mathbb{RP}^2)\) is \(\text{RCD}(0, 3)\) but cannot be a pmGH limit of Riem. manifolds with Ricci \(\geq 0\) and dimension \(\leq 3\).
  If it were, it would be a non-collapsed 3-dimensional Ricci limit. From recent work by Simon and Simon-Topping (idea: mollify by Ricci flow) any non-collapsed 3-dim Ricci limit is a topological manifold, but \(C(\mathbb{RP}^2)\) is not.
Examples of $RCD(K, N)$ spaces

- Ricci limits (i.e. pmGH limits of Riem. Man. with $\text{Ricci} \geq K$ and $\text{dim} \leq N$), no matter if collapsed or not are $RCD(K, N)$
- Finite dimensional Alexandrov spaces with curvature bounded below (Perelman 90’ies and Otsu-Shioya ’94: $Ch$ is quadratic, Petrunin ’12: CD is satisfied)
- Weighted Riemannian manifolds with Bakry-Émery $N - \text{Ricci} \geq K$: i.e. $(M^n, g)$ Riemannian manifold, let $m := \Psi \text{vol}_g$ for some smooth function $\Psi \geq 0$, then $\text{Ric}_{g, \Psi, N} := \text{Ric}_g - (N - n) \frac{\nabla^2 \Psi^{1/N-n}}{\Psi^{1/N-n}} \geq Kg$
  iff $(M, d_g, m)$ is $RCD(K, N)$.
- Cones or spherical suspensions over $RCD(N - 1, N)$spaces (Ketterer)
- Quotients, orbifolds, metric-measure foliations with Ricci bounded below (GalazGarcia-Kell-M.-Sosa).
- Stratified spaces with Ricci bounded below and cone angle $\leq 2\pi$ (Bertrand-Ketterer-Mondello-Richard).
In doing Riemannian geometry one naturally encounters non-smooth spaces

- when taking limits of Riemannian manifolds (procedure used often, e.g. contradiction arguments, blow up arguments, singularities in geometric flows),
- when taking quotients, cones, foliations of Riemannian manifolds.

If the smooth spaces we started with have Ricci bounded below, then the non-smooth spaces arising are $RCD$. $\rightarrow \ RCD(K, N)$ spaces can be seen as an extension of the class of smooth Riemannian manifolds with Ricci $\geq K$, which is closed under many natural geometric and analytic operations. In lecture 4 we will see some smooth applications.
Bochner inequality in m.m.s. setting

- We say that \((X, d, m)\) has the Sobolev-to-Lipschitz property (StL for short) if
  \[ \forall f \in W^{1,2}(X), \quad |\nabla f|^2_w \leq 1 \Rightarrow f \text{ has a } 1\text{-Lipschitz repres.} \]

- \(RCD(K, \infty)\) implies StL (Ambrosio-Gigli-Savaré ’11).

- We say that \((X, d, m)\) satisfies the dimensional Bochner Inequality, \(BI(K, N)\) for short, if
  - it is inf. Hilbert. & StL holds,
  - \(\forall f \in W^{1,2}(X, d, m)\) with \(\Delta f \in L^2(X, m)\) and \(\forall \psi \in \text{LIP}(X)\) with \(\Delta \psi \in L^\infty(X, m)\) it holds

\[
\int_X \left[ \frac{1}{2} |\nabla f|^2_w \Delta \psi + \Delta f \text{ div}(\psi \nabla f) \right] \, dm \geq K \int_X |\nabla f|^2_w \psi \, dm + \frac{1}{N} \int_X |\Delta f|^2 \psi \, dm.
\]

**Question:** are \(RCD^*(K, N)\) and \(BI(K, N)\) equivalent?
Motivation from the smooth setting

- $(M, g)$ smooth Riemannian manifold, $f \in C^\infty(M)$ then Bochner identity

\[
\frac{1}{2} \Delta |\nabla f|^2 = |\text{Hess } f|^2 + \text{Ric}(\nabla f, \nabla f) + g(\nabla \Delta f, \nabla f).
\]

- If $\text{dim}(M) \leq N$ and $\text{Ric} \geq K g$ then Dimensional Bochner inequality

\[
\frac{1}{2} \Delta |\nabla f|^2 \geq \frac{1}{N} |\Delta f|^2 + K |\nabla f|^2 + g(\nabla \Delta f, \nabla f).
\]

To formally obtain $BI(K, N)$ multiply by $\psi \in C_c^\infty(M)$ and integrate by part.

$BI(K, N)$ is a fundamental tool: Splitting theorem of Cheeger-Gromoll 1971, Lichnerowicz bound on spectral gap, upper bound on first Betti number by Bochner via Hodge theory, etc.
**THM** (Erbar-Kuwada-Sturm and Ambrosio-M.-Savaré) $(X, d, m)$ satisfies the dimensional Bochner inequality $BI(K, N)$ iff it is an $RCD^*(K, N)$ space.

- the result bridges the Eulerian formulation of Ricci bounds (Bochner inequality) and the Lagrangian formulation (optimal transport)
- the approach of EKS is based on the equivalence of an entropic curvature condition involving the Boltzman entropy and uses a weighted heat flow (which is linear)
- the (subsequent and independent) proof by AMS involves non linear diffusion equations in metric spaces: more precisely the porous media equation (which is the nonlinear gradient flow of the Renyi entropy) plays a crucial role in the arguments
- the case $N = \infty$ was already established by Ambrosio-Gilgli-Savaré ’12 via the heat flow
Some ideas of the AMS approach in case $K = 0$

**Idea:** Use gradient flows as a bridge between the Eulerian point of view (Bochner inequality) and the Lagrangian point of view (optimal transport).

**Def:** $(\mu_t)_{t \geq 0} \subset \mathcal{P}_2(X)$ is an *EVI*$_0$ gradient flow of $\mathcal{U}_{N,m}$ starting from $\mu_0 \in \mathcal{P}_2(X)$ if

$$\frac{1}{2} \frac{d}{dt} W_2^2(\mu_t, \nu) \leq \mathcal{U}_{N,m}(\nu) - \mathcal{U}_{N,m}(\mu_t), \quad \forall \nu \in \mathcal{P}(X)$$

The two parts of the bridge are:

**Thm 1:** [Ambrosio-M.-Savaré] $(X, d, m)$ satisfies the *RCD*$_*(0, N)$ condition iff for every $\mu_0 \in \mathcal{P}_2(X)$, the Renyi entropy $\mathcal{U}_{N,m}$ admits an *EVI*$_0$ gradient flow starting from $\mu_0$.

**Thm 2:** [Ambrosio-M.-Savaré]. $(X, d, m)$ satisfies the *BL*(0, N) condition iff for every $\mu_0 \in \mathcal{P}_2(X)$, the Renyi entropy $\mathcal{U}_{N,m}$ admits an *EVI*$_0$ gradient flow starting from $\mu_0$. 
Formal proof of $BI(0, N) \Rightarrow EVI_0$

Inspired by smooth setting (Otto-Westdickenberg and Daneri-Savaré) consider the porous media flow.

- Let $P_t$ be the porous media flow defined by the equation
  \[
  \partial_t P_t(\rho_0) = \Delta((P_t \rho_0)^{1-\frac{1}{N}})
  \]

- CLAIM: $(P_t \rho_0)m$ defines an $EVI_0$ gradient flow from $\mu_0 = \rho_0m$

- Given a regular curve $(\rho_s m)_{s\in[0,1]} \subset \mathcal{P}(X)$ call $\rho_{s,t} := P_{st}\rho_s$

- For every $s$ and $t$ let $\varphi_{s,t}$ be the solution to the continuity equation $\text{div}(\rho_{s,t} \nabla \varphi_{s,t}) = \partial_s \rho_{s,t}$

- $\nabla \varphi_{s,t}$ has to be understood as the velocity field of the curve $s \mapsto \rho_{s,t}$

Lemma 5 The $BI(0, N)$ condition implies that

\[
\frac{d}{dt} \int_X \frac{1}{2} |\nabla \varphi_{s,t}|^2 \rho_{s,t} \, dm + \frac{d}{ds} \mathcal{U}_{N,m}(\rho_{s,t}m) \leq 0
\]
Thanks to the semigroup property of $P_t$, enough to check that $EVI_0$ holds at $t = 0$, i.e.

$$\frac{1}{2} \frac{d}{dt^+} W^2_2((P_t \mu_0), \nu) \leq \mathcal{U}_{N,m}(\nu) - \mathcal{U}_{N,m}^{\prime}(\mu_0), \forall \nu \in \mathcal{P}(X)$$

Let $(\rho_s m)_{s \in [0,1]}$ be a geodesic from to $\nu = \rho_0 m$ to $\mu_0 : = \rho_1 m$ (by density it is enough to consider $\nu \ll m$)

Integrating Lemma 5 w.r.t. $s \in [0,1]$ we get

$$\frac{d}{dt} \int_0^1 \left( \int_X \frac{1}{2} |\nabla \varphi_{s,t}|^2 \rho_{s,t} \, dm \right) \, ds \leq \mathcal{U}_{N,m}(\nu) - \mathcal{U}_{N,m}(\mu_0)$$
Conclude observing that

\[ W^2_2(P_t(\mu_0), \nu) \leq \int_0^1 \left( \int_X \frac{1}{2} |\nabla \varphi_{s,t}|^2 \rho_{s,t} \,dm \right) \,ds \]

= “Length of \((P_{st}\rho_s)m)_{s\in[0,1]}\) from \(\nu\) to \(P_t\mu_0\)”

\[ W^2_2(\mu_0, \nu) = \int_0^1 \left( \int_X \frac{1}{2} |\nabla \varphi_{s,0}|^2 \rho_s \,dm \right) \,ds \]

= “Length of \((\rho_s m)_{s\in[0,1]}\), \(W_2\)-geod from \(\nu\) to \(\mu_0\)”

so that

\[ \frac{1}{2} \frac{d}{dt^+} W^2_2((P_t\mu_0), \nu) \leq \frac{d}{dt} \int_0^1 \left( \int_X \frac{1}{2} |\nabla \varphi_{s,t}|^2 \rho_{s,t} \,dm \right) \,ds \]
Consequences of Bochner inequality.
1: Li-Yau and Harnack type inequalities

**THM** [Garofalo-M. '13, Jiang '14] \( f \in L^1(X, m), f \geq 0 \text{ m-a.e.} \) Then

- **Li-Yau Inequality**: if \((X, d, m)\) is an \( RCD^*(0, N) \) space then
  \[
  \Delta(\log(H_t f)) \geq -\frac{N}{2t} \quad \text{m-a.e.} \quad \forall t > 0
  \]

- **Bakry-Quian Inequality**: If \((X, d, m)\) is an \( RCD^*(K, N) \) space, for some \( K > 0 \), then
  \[
  \Delta(H_t f) \leq \frac{NK}{4} (H_t f) \quad \text{m-a.e.} \quad \forall t > 0
  \]

- **Harnack Inequality**: If \((X, d, m)\) is an \( RCD^*(K, N) \) space, for some \( K \geq 0 \), then for every \( x, y \in X \) and \( 0 < s < t \) we have
  \[
  (H_t f)(y) \geq (H_s f)(x) e^{-\frac{d^2(x,y)}{4(t-s)e^{\frac{2Ks}{3}}}} \left( \frac{1 - e^{\frac{2K}{3} t}}{1 - e^{\frac{2K}{3} s}} \right)^\frac{N}{2}.
  \]
Localization-Globalization of Curvature-Dimension conditions

Curvature-dimension bounds are geometrically local concepts, but the Lott-Sturm-Villani definition is global in nature. So does global to local and local to global hold?

**GTL**: typically needs some strong convexity either of the entropy or of the domain (Book of Villani)

**LTG**: was established under the non-branching assumption for

- $CD(K, \infty)$ spaces [Sturm '06], $CD(0, N)$ spaces [Villani '09]
- $CD^*(K, N)$ spaces [Bacher-Sturm '10]

**BUT**: i) Non Branch $+ CD(K,N)$ is NOT stable under mGH-conv.
   ii) Rajala '13: example of a (highly branching) locally $CD^*(0, 4) = CD(0, 4)$ space but not $CD(K, \infty)$.

**Q**: how reinforce $CD^*(K, N)$ to get a stable condition $+ LTG$?
Consequences of Bochner inequality.

2: Local to Global property for \( RCD^*(K, N) \) without a priori non-branching assumption

**THM**[Ambrosio-M.-Savaré ’13] Let \((X, d, m)\) be a locally compact length space and assume there is a covering \(\{U_i\}_{i \in I}\) of \(X\) by non empty open subsets s.t. \((\bar{U}_i, d, m_{\bar{U}_i})\) satisfy \(RCD(K, \infty)\) (resp. \(RCD^*(K, N)\)). Then \((X, d, m)\) is an \(RCD(K, \infty)\) (resp. \(RCD^*(K, N)\)) space.

**IDEA of PROOF**

(i) by equivalence with \(BI(K, N)\), for every \(U_i\) the dimensional Bochner inequality holds for functions supported on \(U_i\)

(ii) Construct partition of unity \(\{\chi_i\}_{i \in I}\) subordinated to \(\{U_i\}_{i \in I}\) of Lipschitz functions with \(\Delta \chi_i \in L^\infty\)

(iii) Globalize \(BI(K, N)\) by using partition of unity and conclude that \(RCD^*(K, N)\) holds by applying globally the equivalence theorem
DEF (Rajala-Sturm ’14): $(X, d, m)$ is essentially non-branching (e.n.b. for short) if for every two measures $\mu_0, \mu_1 \in P_2(X)$, $\mu_0, \mu_1 \ll m$ the $W_2$ optimal transport concentrates on non-branching geodesics.

THM (Rajala-Sturm ’14): Any $RCD(K, \infty)$ or $RCD^*(K, N)$ space is essentially non-branching.

THM (Cavalletti-Milman ’17) Let $(X, d, m)$ be an e.n.b. space with $m(X) < \infty$. Then

$$CD^*(K, N) \Leftrightarrow CD^*_{loc}(K, N) \Leftrightarrow CD_{loc}(K, N) \Leftrightarrow CD(K, N).$$

In particular $RCD^*(K, N) \Leftrightarrow RCD(K, N)$.

The proof uses localization techniques and $L^1$-optimal transport. Some hints of these techniques in the next lectures.

For simplicity of notation, from now on we will write $RCD(K, N)$ in place of $RCD^*(K, N)$, but all the results hold in $RCD^*(K, N)$. 