Metric measure spaces satisfying Ricci curvature lower bounds
Lecture 2

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Summer School 2021
Curvature Constraints and Spaces of Metrics

June 14–July 2, 2021
Institut Fourier-Grenoble (hybrid)

European Research Council
Established by the European Commission

Recall from Lecture 1: Def of $CD(K, N)$

- $CD(K, N)$ condition [Lott-Sturm-Villani '06]: fixed $N \in [1, +\infty]$ and $K \in \mathbb{R}$, $(X, \mathsf{d}, \mathfrak{m})$ is a $CD(K, N)$-space if the Entropy $\mathcal{U}_{N,\mathfrak{m}}$ is K-convex along geodesics in $(\mathcal{P}_2(X), W_2)$ (for finite N is a "distorted" (K, N)-geod. conv.).
  Cases easier to write, to keep in mind:
  - $CD(K, \infty)$: for every $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ there exists a W_2-geodesic $(\mu_t)_{t\in[0,1]}$ such that for every $t \in [0, 1]$ it holds

$$\mathcal{U}_{\infty,\mathfrak{m}}(\mu_t) \leq (1-t)\mathcal{U}_{\infty,\mathfrak{m}}(\mu_0)+t\mathcal{U}_{\infty,\mathfrak{m}}(\mu_1)-\frac{K}{2}t(1-t)W_2(\mu_0, \mu_1)^2.$$

  - $CD(0, N)$: for every $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ there exists a W_2-geodesic $(\mu_t)_{t\in[0,1]}$ such that for every $t \in [0, 1]$ it holds

$$\mathcal{U}_{N,\mathfrak{m}}(\mu_t) \leq (1 - t)\mathcal{U}_{N,\mathfrak{m}}(\mu_0) + t\mathcal{U}_{N,\mathfrak{m}}(\mu_1).$$

Recall from Lecture 1: stability of $CD(K, N)$

- THM: Fix $K \in \mathbb{R}$ and $N \in [1, \infty]$. Let $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$, $n \in \mathbb{N}$, be a sequence of $CD(K, N)$ p.m.m. spaces converging to $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ in the pmGH-sense (or, more generally, in the pmG sense).
  Then $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty)$ is a $CD(K, N)$ space as well.
- From the stability of $CD(K, N)$: any pmGH limit of a sequence of Riemannian manifolds with Ricci $\geq K$ and dim $\leq N$ is a $CD(K, N)$ space. $\rightsquigarrow$ $CD(K, N) \supset$ Ricci limits.

Plan of Lecture 2

- Introduce $RCD(K, \infty)$
- Discuss gradient flows formulation of $RCD(K, \infty)$
- Finite dimensional theory: $RCD(K, N)$ and $RCD^*(K, N)$
- Bochner inequality and some applications

Part 1: $RCD(K, \infty)$ spaces

$CD(K,N)$ Vs Ricci limits

- From the stability of $CD(K,N)$: any pmGH limit of a sequence of Riemannian manifolds with Ricci $\geq K$ and dim $\leq N$ is a $CD(K,N)$ space. $\rightsquigarrow$ $CD(K,N) \supset$ Ricci limits.
- Finsler manifolds with Ricci curvature bounded below are $CD(K,N)$. E.g. $(\mathbb{R}^N, \|\cdot\|, \mathcal{L}^N)$ is $CD(0,N)$, $\|\cdot\|$ any norm.
- FACT: If a smooth Finsler manifold M is a Ricci-limit space then M is Riemannian (Cheeger-Colding '00).$\rightsquigarrow$ the class of $CD(K,N)$ is, in some sense, too large.
- Moreover, and maybe more importantly, some fundamental theorems in comparison geometry of Riemannian manifolds (e.g. Cheeger-Gromoll Splitting Theorem) are not true in the larger Finsler category (e.g. $(\mathbb{R}^2, \|\cdot\|_\infty)$ is $CD(0,2)$, contains a line but does not split isometrically).
- $\rightsquigarrow$ We wish to reinforce the $CD(K,N)$ condition in order to isolate the "Riemannian" $CD(K,N)$ spaces; in other words, we wish to rule out Finsler structures, but in a sufficiently weak way to still get a STABLE notion under pmGH converg.

Cheeger energy and $RCD(K, \infty)$ spaces

- Given a m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ and $f \in L^2(X, \mathfrak{m})$, define the Cheeger energy

$$Ch_{\mathfrak{m}}(f) := \frac{1}{2} \int_X |\nabla f|_w^2 \, d\mathfrak{m} = \liminf_{u \to f \; in L^2} \frac{1}{2} \int_X (\mathrm{lip}\, u)^2 d\mathfrak{m}$$

  where $|\nabla f|_w$ is the minimal weak upper gradient.
- Crucial observation: On a Finsler manifold M, the Cheeger energy is quadratic (i.e. parallelogram identity holds) iff M is Riemannian. A m.m.s. where the Cheeger energy is quadratic is said infinitesimally Hilbertian.
- Idea Reinforce the CD condition by asking that the Cheeger energy is quadratic.

DEF(Ambrosio-Gigli-Savaré, see also Ambrosio-Gigli-M.-Rajala)
Given $K \in \mathbb{R}$, $(X, \mathsf{d}, \mathfrak{m})$ is an $RCD(K, \infty)$ space if it is a $CD(K, \infty)$ space & infinitesimally Hilbertian.
Question: is $RCD(K, \infty)$ stable under pmG-convergence?

Stability of heat flow under pmG-convergence

- $Ch_{\mathfrak{m}} : L^2(X, \mathfrak{m}) \to \mathbb{R}$ is a convex and l.s.c. functional so (by classical theory of gradient flows, e.g. Brezis) admits a unique gradient flow $(H_t)_{t \geq 0}$ called Heat flow.
- If $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n) \to (X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ in the pmG-sense, then there is a way to define convergence of a sequence $f_n \in L^2(X_n, \mathfrak{m}_n)$ to a function $f_\infty \in L^2(X_\infty, \mathfrak{m}_\infty)$

THM(Gigli '11-Gigli-M-Savaré '13)[Stability of Heat flows]
Let $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n) \to (X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ in the pmG-sense, X_n are $CD(K, \infty)$-spaces. If $f_n \in L^2(X_n, \mathfrak{m}_n)$ strongly L^2-converges to $f_\infty \in L^2(X_\infty, \mathfrak{m}_\infty)$, then

$$H_t^n(f_n) \to H_t^\infty(f_\infty) \text{ strongly in } L^2 \text{ for every } t \geq 0.$$

Idea of proof: i) Mosco convergence of Cheeger energies under pmG-convergence (pass via the entropy).
ii) convergence of resolvant maps
iii) approximate the heat flow by iterated resolvant maps to conclude. □

Stability of $RCD(K,\infty)$ under pmG-convergence

Fact: $(X, \mathrm{d}, \mathfrak{m})$ is infinitesimally Hilbertian iff $H_t : L^2(X, \mathfrak{m}) \to L^2(X, \mathfrak{m})$ is linear for every $t > 0$.

THM (Ambrosio-Gigli-Savaré '11, Gigli-M-Savaré '13): Let $(X_n, \mathrm{d}_n, \mathfrak{m}_n, \bar{x}_n)$, $n \in \mathbb{N}$, be a sequence of $RCD(K,\infty)$ p.m.m. spaces converging to a limit space $(X_\infty, \mathrm{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ in the pmG-sense. Then $(X_\infty, \mathrm{d}_\infty, \mathfrak{m}_\infty)$ is $RCD(K,\infty)$ as well.

Idea of proof:
i) we already know that $CD(K,\infty)$ is stable, so $(X_\infty, \mathrm{d}_\infty, \mathfrak{m}_\infty)$ is a $CD(K,\infty)$ space.
ii) since the heat flows of X_n are linear, by the stability of heat flows also the limit heat flow is linear.

□

$RCD(K, \infty)$ via gradient flows. Preliminaries

- Let (Y, d_Y) be a geodesic space (later we will take $(Y, \mathrm{d}_Y) = (\mathcal{P}_2(X), W_2))$. A functional $E : Y \to \mathbb{R} \cup \{+\infty\}$ is said *K-convex* if for every $y_0, y_1 \in Y$ there exists a constant speed geodesic $\gamma : [0,1] \to Y$ such that

$$\gamma_0 = y_0 \quad \& \quad \gamma_1 = y_1 \qquad \text{and}$$

$$E(\gamma_t) \leq (1-t)E(y_0) + tE(y_1) - \frac{K}{2} t(1-t)\mathrm{d}_Y^2(y_0, y_1)$$

- f smooth K-convex function on $\mathbb{R}^n$, then $v = -\nabla f(x)$ iff

$$< v, x - y > + \frac{K}{2}|x-y|^2 + f(x) \leq f(y) \quad \forall y \in \mathbb{R}^n$$

so (x_t) is a gradient flow of f, i.e. $x_t' = -\nabla f(x)$ iff

$$\frac{d}{dt}\frac{|x_t - y|^2}{2} + \frac{K}{2}|x_t - y|^2 + f(x_t) \leq f(y) \quad \forall y \in \mathbb{R}^n, t \geq 0.$$

$RCD(K,\infty)$ via gradient flows. The EVI_K condition

- Definition of EVI_K flow: Let (Y, d_Y) be a geodesic space and E a l.s.c. functional. A locally absolutely continuous curve (y_t) with $y_t \in D(E)$ for every $t > 0$ is said EVI_K gradient flow of E if

$$\frac{d}{dt}\frac{\mathsf{d}^2_Y(y_t, z)}{2} + \frac{K}{2}\mathsf{d}^2_Y(y_t, z) + E(y_t) \leq E(z) \quad \forall z \in Y, a.e.t > 0.$$

- Remark: the existence of an EVI_K flow of E depends on both the K-convexity of E and the infinitesimal Hilbertianity of the space; existence is not true in general, but in case of existence then nice contractivity and regularizing properties of the flow hold.

$RCD(K,\infty)$ is equivalent to EVI_K

Theorem[Ambrosio-Gigli-Savaré(2011)-Ambrosio-Gigli-M.-Rajala(2012)]: $(X, \mathrm{d}, \mathfrak{m})$ is $RCD(K,\infty)$ iff for every $\mu \in \mathcal{P}_2(X)$ with $supp(\mu) \subset supp(\mathfrak{m})$ there exists an EVI_K gradient flow (μ_t) of $\mathcal{U}_{\infty,\mathfrak{m}}$ in $(\mathcal{P}_2(X), W_2)$ starting from μ.

Remark

- EVI_K formulation useful for proving the stability of $RCD(K,\infty)$ [Ambrosio-Gigli-Savaré(2011)] for $\mathfrak{m} \in \mathcal{P}(X)$ and [Gigli-M.-Savaré(2013)] for general $\mathfrak{m}$ and without any compactness assumption.
- EVI_K formulation useful to prove that $RCD(K,\infty)$ is equivalent to $BE(K,\infty)$, i.e. Bochner inequality for $N=\infty$ [Ambrosio-Gigli-Savaré(2012)].
  (See later for more on Bochner inequality)

Finite dimensional Theory: $RCD^*(K, N)$ and $RCD(K, N)$ spaces

The $CD^*(K,N)$ condition

- The reduced curvature-dimension condition $CD^*(K,N)$ was introduced by Bacher-Sturm (2010)
- Modification of $CD(K,N)$: (a priori) weaker convexity condition on $\mathcal{U}_{N,\mathfrak{m}}$
- $CD(K,N) \Rightarrow CD^*(K,N) \Rightarrow CD(K^*,N)$ where $K^* = \frac{K(N-1)}{N}$
- If (X,d) is non branching then (local to global) $CD^*(K,N) \Leftrightarrow CD^*_{loc}(K,N) \Leftrightarrow CD_{loc}(K^-,N)$.
- In non branching spaces tensorization holds
- Same geometric consequence of $CD(K,N)$ (Bishop-Gromov, Bonnet-Myers, Lichnerowicz) but sometimes with slightly worse constants.

$RCD^*(K,N)$ and $RCD(K,N)$ spaces

DEF: $(X, \mathsf{d}, \mathfrak{m})$ is $RCD^*(K,N)$ (resp. $RCD(K,N)$) iff Infinit. Hilbertian $CD^*(K,N)$ (resp. $CD(K,N)$).

Stability: If $(X_i, \mathsf{d}_i, \mathfrak{m}_i, \bar{x}_i)$ are $RCD^*(K,N)$ (resp. $RCD(K,N)$) and converge in pmGH (or pmG) to $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$, then $(X, \mathsf{d}, \mathfrak{m})$ is $RCD^*(K,N)$ (resp. $RCD(K,N)$) as well.

$\rightsquigarrow$ any pmGH limit of a sequence of Riemannian manifolds with Ricci $\geq K$ and dim $\leq N$ is a $RCD(K,N)$ space

$RCD(K,\infty) \supset RCD^*(K,N) \supset RCD(K,N) \supset$ Ricci limits of N-dim man

for every $N \in \mathbb{N}$.

Q: are the inclusions above strict or not?

About $RCD(K,\infty) \supset RCD^*(K,N) \supset RCD(K,N) \supset$ Ricci limits

- $RCD(K,\infty) \supsetneq RCD^*(K,N)$.
  Example: Gaussian space $(\mathbb{R}^n, \|\cdot\|_{\mathrm{eucl}}, \gamma_n)$
- THM (Cavalletti-Milman, Inventiones Math. 2021): In case $\mathfrak{m}(X) < \infty$ it holds $RCD^*(K,N) \Leftrightarrow RCD(K,N)$.
  No reason to expect if fails for $\mathfrak{m}(X) = \infty$ but adaptation of proof is not trivial.
- $RCD(K,N) \supsetneq$ pmGH limits of Riem. man. with Ricci $\geq K$ and dimension $\leq N$.
  Example: $C(\mathbb{RP}^2)$ is $RCD(0,3)$ but cannot be a pmGH limit of Riem. manifolds with Ricci ≥ 0 and dimension ≤ 3.
  If it were, it would be a non-collapsed 3-dimensional Ricci limit. From recent work by Simon and Simon-Topping (idea: mollify by Ricci flow) any non-collapsed 3-dim Ricci limit is a topological manifold, but $C(\mathbb{RP}^2)$ is not.

Examples of $RCD(K, N)$ spaces

- Ricci limits (i.e. pmGH limits of Riem. Man. with Ricci$\geq K$ and dim$\leq N$), no matter if collapsed or not are $RCD(K, N)$
- Finite dimensional Alexandrov spaces with curvature bounded below (Perelman 90'ies and Otsu-Shioya '94: Ch is quadratic, Petrunin '12: CD is satisfied)
- Weighted Riemannian manifolds with Bakry-Émery $N - Ricci \geq K$: i.e. (M^n, g) Riemannian manifold, let $\mathfrak{m} := \Psi\, vol_g$ for some smooth function $\Psi \geq 0$, then $Ric_{g,\Psi,N} := Ric_g - (N-n)\frac{\nabla^2 \Psi^{1/N-n}}{\Psi^{1/N-n}} \geq Kg$ iff $(M, \mathsf{d}_g, \mathfrak{m})$ is $RCD(K, N)$.
- Cones or spherical suspensions over $RCD(N-1, N)$spaces (Ketterer)
- Quotients, orbifolds, metric-measure foliations with Ricci bounded below (GalazGarcia-Kell-M.-Sosa).
- Stratified spaces with Ricci bounded below and cone angle$\leq 2\pi$ (Bertrand-Ketterer-Mondello-Richard).

Moral from examples

In doing Riemannian geometry one naturally encounters non smooth spaces

- when taking limits of Riemannian manifolds (procedure used often, e.g. contradiction arguments, blow up arguments, singularities in geometric flows),
- when taking quotients, cones, foliations of Riemannian manifolds.

If the smooth spaces we started with have Ricci bounded below, then the non smooth spaces arising are RCD.

$\rightarrow$ $RCD(K, N)$ spaces can be seen as an extension of the class of smooth Riemannian manifolds with Ricci $\geq K$, which is closed under many natural geometric and analytic operations.
In lecture 4 we will see some smooth applications.

Bochner inequality in m.m.s. setting

- We say that $(X, \mathrm{d}, \mathfrak{m})$ has the Sobolev-to-Lipschitz property (StL for short) if
  $\forall f \in W^{1,2}(X),\ |\nabla f|_w^2 \leq 1 \Rightarrow f$ has a 1-Lipschitz repres.
- $RCD(K, \infty)$ implies StL (Ambrosio-Gigli-Savaré '11).
- We say that $(X, \mathrm{d}, \mathfrak{m})$ satisfies the dimensional Bochner Inequality, $BI(K, N)$ for short, if
  -it is inf. Hilbert. & StL holds,
  -$\forall f \in W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ with $\triangle f \in L^2(X, \mathfrak{m})$ and $\forall \psi \in LIP(X)$ with $\triangle \psi \in L^\infty(X, \mathfrak{m})$ it holds

$$\int_X \left[\frac{1}{2}|\nabla f|_w^2 \Delta\psi + \Delta f \, div(\psi \nabla f)\right] d\mathfrak{m} \quad \geq \quad K \int_X |\nabla f|_w^2 \psi d\mathfrak{m} + \frac{1}{N}\int_X |\Delta f|^2 \psi d\mathfrak{m}.$$

Question: are $RCD^*(K, N)$ and $BI(K, N)$ equivalent?

Motivation from the smooth setting

- (M, g) smooth Riemannian manifold, $f \in C^\infty(M)$ then Bochner identity

$$\frac{1}{2}\Delta|\nabla f|^2 = |Hess\, f|^2 + Ric(\nabla f, \nabla f) + g(\nabla \Delta f, \nabla f).$$

- If $dim(M) \leq N$ and $Ric \geq K\, g$ then Dimensional Bochner inequality

$$\frac{1}{2}\Delta|\nabla f|^2 \geq \frac{1}{N}|\Delta f|^2 + K|\nabla f|^2 + g(\nabla \Delta f, \nabla f).$$

To formally obtain $BI(K, N)$ multiply by $\psi \in C_c^\infty(M)$ and integrate by part.

$BI(K, N)$ is a fundamental tool: Splitting theorem of Cheeger-Gromoll 1971, Lichnerowictz bound on spectral gap, upper bound on first Betti number by Bochner via Hodge theory, etc.

$RCD^*(K,N)$ is equivalent to $BI(K,N)$

THM(Erbar-Kuwada-Sturm and Ambrosio-M.-Savaré)
$(X, \mathrm{d}, \mathfrak{m})$ satisfies the dimensional Bochner inequality $BI(K,N)$ iff it is an $RCD^*(K,N)$ space.

- the result bridges the Eulerian formulation of Ricci bounds (Bochner inequality) and the Lagrangian formulation (optimal transport)
- the approach of EKS is based on the equivalence of an entropic curvature condition involving the Boltzman entropy and uses a weighted heat flow (which is linear)
- the (subsequent and independent) proof by AMS involves non linear diffusion equations in metric spaces: more precisely the porous media equation (which is the nonlinear gradient flow of the Renyi entropy) plays a crucial role in the arguments
- the case $N = \infty$ was already established by Ambrosio-Gilgli-Savaré '12 via the heat flow

Some ideas of the AMS approach in case $K = 0$

Idea: Use gradient flows as a bridge between the Eulerian point of view (Bochner inequality) and the Lagrangian point of view (optimal transport).

Def: $(\mu_t)_{t\geq 0} \subset \mathcal{P}_2(X)$ is an *EVI_0* gradient flow of $\mathcal{U}_{N,\mathfrak{m}}$ starting from $\mu_0 \in \mathcal{P}_2(X)$ if

$$\frac{1}{2}\frac{d}{dt} W_2^2(\mu_t, \nu) \leq \mathcal{U}_{N,\mathfrak{m}}(\nu) - \mathcal{U}_{N,\mathfrak{m}}(\mu_t), \ \forall \nu \in \mathcal{P}(X)$$

The two parts of the bridge are:

Thm 1:[Ambrosio-M.-Savaré] $(X, \mathrm{d}, \mathfrak{m})$ satisfies the $RCD^*(0, N)$ condition iff for every $\mu_0 \in \mathcal{P}_2(X)$, the Renyi entropy $\mathcal{U}_{N,\mathfrak{m}}$ admits an EVI_0 gradient flow starting from μ_0.

Thm 2:[Ambrosio-M.-Savaré]. $(X, \mathrm{d}, \mathfrak{m})$ satisfies the $BI(0, N)$ condition iff for every $\mu_0 \in \mathcal{P}_2(X)$, the Renyi entropy $\mathcal{U}_{N,\mathfrak{m}}$ admits an EVI_0 gradient flow starting from μ_0.

Formal proof of $BI(0, N) \Rightarrow EVI_0$

Inspired by smooth setting (Otto-Westdickenberg and Daneri-Savaré) consider the porous media flow.

- Let P_t be the porous media flow defined by the equation

$$\partial_t P_t(\rho_0) = \Delta((P_t\rho_0)^{1-\frac{1}{N}})$$

- CLAIM:$(P_t\rho_0)\mathfrak{m}$ defines an EVI_0 gradient flow from $\mu_0 = \rho_0\mathfrak{m}$
- Given a regular curve $(\rho_s\mathfrak{m})_{s\in[0,1]} \subset \mathcal{P}(X)$ call $\rho_{s,t} := P_{st}\rho_s$
- For every s and t let $\varphi_{s,t}$ be the solution to the continuity equation $div(\rho_{s,t}\nabla\varphi_{s,t}) = \partial_s\rho_{s,t}$
- $\nabla\varphi_{s,t}$ has to be understood as the velocity field of the curve $s \mapsto \rho_{s,t}$

Lemma 5 The $BI(0, N)$ condition implies that

$$\frac{d}{dt}\int_X \frac{1}{2}|\nabla\varphi_{s,t}|^2\rho_{s,t}\,d\mathfrak{m} + \frac{d}{ds}\mathcal{U}_{N,\mathfrak{m}}(\rho_{s,t}\mathfrak{m}) \leq 0$$

Formal proof of $BI(0, N) \Rightarrow EVI_0$

- Thanks to the semigroup property of P_t, enough to check that EVI_0 holds at $t = 0$, i.e.

$$\frac{1}{2}\frac{d}{dt^+}W_2^2((P_t\mu_0), \nu) \leq \mathcal{U}_{N,\mathfrak{m}}(\nu) - \mathcal{U}_{N,\mathfrak{m}}(\mu_0), \ \forall \nu \in \mathcal{P}(X)$$

- Let $(\rho_s \mathfrak{m})_{s\in[0,1]}$ be a geodesic from to $\nu = \rho_0 \mathfrak{m}$ to $\mu_0 := \rho_1 \mathfrak{m}$ (by density it is enough to consider $\nu \ll \mathfrak{m}$)
- Integrating Lemma 5 w.r.t. $s \in [0, 1]$ we get

$$\frac{d}{dt}\int_0^1 \left(\int_X \frac{1}{2} |\nabla \varphi_{s,t}|^2 \rho_{s,t} \, d\mathfrak{m} \right) ds \leq \mathcal{U}_{N,\mathfrak{m}}(\nu) - \mathcal{U}_{N,\mathfrak{m}}(\mu_0)$$

Formal proof of $BI(0, N) \Rightarrow EVI_0$. Conclusion

Conclude observing that

$$\begin{aligned}
W_2^2(P_t(\mu_0), \nu) &\leq \int_0^1 \left(\int_X \frac{1}{2} |\nabla \varphi_{s,t}|^2 \rho_{s,t} \, d\mathfrak{m} \right) ds \\
&= \text{"Length of } ((P_{st}\rho_s)\mathfrak{m})_{s \in [0,1]} \text{ from } \nu \text{ to } P_t\mu_0\text{"} \\
W_2^2(\mu_0, \nu) &= \int_0^1 \left(\int_X \frac{1}{2} |\nabla \varphi_{s,0}|^2 \rho_s \, d\mathfrak{m} \right) ds \\
&= \text{"Length of } (\rho_s \mathfrak{m})_{s \in [0,1]}, \, W_2\text{-geod from } \nu \text{ to } \mu_0\text{"}
\end{aligned}$$

so that

$$\frac{1}{2} \frac{d}{dt^+} W_2^2((P_t\mu_0), \nu) \leq \frac{d}{dt} \int_0^1 \left(\int_X \frac{1}{2} |\nabla \varphi_{s,t}|^2 \rho_{s,t} \, d\mathfrak{m} \right) ds$$

Consequences of Bochner inequality.
1: Li-Yau and Harnack type inequalities

THM[Garofalo-M. '13,Jiang '14] $f \in L^1(X, \mathfrak{m})$,$f \geq 0$ $\mathfrak{m}$-a.e.Then

- Li-Yau Inequality: if $(X, \mathsf{d}, \mathfrak{m})$ is an $RCD^*(0, N)$ space then
$$\Delta(\log(H_t f)) \geq -\frac{N}{2t} \quad \mathfrak{m}\text{-a.e.} \quad \forall t > 0$$
- Bakry-Quian Inequality: If $(X, \mathsf{d}, \mathfrak{m})$ is an $RCD^*(K, N)$ space, for some $K > 0$, then
$$\Delta(H_t f) \leq \frac{NK}{4}(H_t f) \quad \mathfrak{m}\text{-a.e.} \quad \forall t > 0$$
- Harnack Inequality: If $(X, \mathsf{d}, \mathfrak{m})$ is an $RCD^*(K, N)$ space, for some $K \geq 0$, then for every $x, y \in X$ and $0 < s < t$ we have
$$(H_t f)(y) \geq (H_s f)(x)\, e^{-\frac{\mathsf{d}^2(x,y)}{4(t-s)e^{\frac{2Ks}{3}}}} \left(\frac{1 - e^{\frac{2K}{3}s}}{1 - e^{\frac{2K}{3}t}} \right)^{\frac{N}{2}}.$$

Localization-Globalization of Curvature-Dimension conditions

Curvature-dimension bounds are geometrically local concepts, but the Lott-Sturm-Villani definition is global in nature. So does global to local and local to global hold?

GTL: typically needs some strong convexity either of the entropy or of the domain (Book of Villani)

LTG: was established under the non-branching assumption for

- $CD(K, \infty)$ spaces [Sturm '06], $CD(0, N)$ spaces [Villani '09]
- $CD^*(K, N)$ spaces [Bacher-Sturm '10]

BUT: i) Non Branch + CD(K,N) is NOT stable under mGH-conv.
ii) Rajala '13: example of a (highly branching) locally $CD^*(0, 4) = CD(0, 4)$ space but not $CD(K, \infty)$.
Q: how reinforce $CD^*(K, N)$ to get a stable condition + LTG?

Consequences of Bochner inequality. 2: Local to Global property for $RCD^*(K,N)$ without a priori non-branching assumption

THM[Ambrosio-M.-Savaré '13] Let $(X, \mathsf{d}, \mathfrak{m})$ be a locally compact length space and assume there is a covering $\{U_i\}_{i \in I}$ of X by non empty open subsets s.t. $(\bar{U}_i, \mathsf{d}, \mathfrak{m} \llcorner \bar{U}_i)$ satisfy $RCD(K, \infty)$ (resp. $RCD^*(K,N)$).
Then $(X, \mathsf{d}, \mathfrak{m})$ is an $RCD(K, \infty)$ (resp. $RCD^*(K,N)$) space.

IDEA of PROOF
(i) by equivalence with $BI(K,N)$, for every U_i the dimensional Bochner inequality holds for functions supported on U_i
(ii) Construct partition of unity $\{\chi_i\}_{i \in I}$ subordinated to $\{U_i\}_{i \in I}$ of Lipschitz functions with $\Delta \chi_i \in L^\infty$
(iii) Globalize $BI(K,N)$ by using partition of unity and conclude that $RCD^*(K,N)$ holds by applying globally the equivalence theorem □

Local to global for Essentially Non Branching $CD(K,N)$

DEF(Rajala-Sturm '14): $(X,\mathsf{d},\mathfrak{m})$ is essentially non-branching (e.n.b. for short) if for every two measures $\mu_0,\mu_1 \in \mathcal{P}_2(X)$, $\mu_0,\mu_1 \ll \mathfrak{m}$ the W_2 optimal transport concentrates on non-branching geodesics.

THM (Rajala-Sturm '14): Any $RCD(K,\infty)$ or $RCD^*(K,N)$ space is essentially non-branching.

THM (Cavalletti-Milman '17) Let $(X,\mathsf{d},\mathfrak{m})$ be an e.n.b. space with $\mathfrak{m}(X)<\infty$. Then

$$CD^*(K,N) \Leftrightarrow CD^*_{loc}(K,N) \Leftrightarrow CD_{loc}(K,N) \Leftrightarrow CD(K,N).$$

In particular $RCD^*(K,N) \Leftrightarrow RCD(K,N)$.

The proof uses localization techniques and L^1-optimal transport. Some hints of these techniques in the next lectures.

For simplicity of notation, from now on we will write $RCD(K,N)$ in place of $RCD^*(K,N)$, but all the results hold in $RCD^*(K,N)$.