Metric measure spaces satisfying Ricci curvature lower bounds
Lecture 3

Andrea Mondino (University of Oxford)

Summer School 2021
Curvature Constraints and Spaces of Metrics

June 14–July 2, 2021
Institut Fourier-Grenoble (hybrid)

European Research Council
Established by the European Commission

Recap

- Lecture 1: $CD(K, N)$ spaces, $K \in \mathbb{R}$, $N \in [1, +\infty]$.
  Motivation, definition, stability.
- Lecture 2: $RCD \& RCD^*(K, N)$ spaces, $K \in \mathbb{R}$, $N \in [1, +\infty]$.
  Motivation, definition, stability, Bochner inequality.
- Lecture 3: structure theory of $RCD^*(K, N)$ spaces,
  $K \in \mathbb{R}$, $N \in [1, +\infty)$.
- Lecture 4: sharp & rigid geometric & functional inequalities

Splitting Theorem

A key theorem for smooth Riemannian manifolds is the

Splitting Theorem (Cheeger-Gromoll '71) Let (M, g) be a complete smooth Riemannian manifold with Ricci≥ 0. Assume M contains a line, i.e. an isometric immersion of $\mathbb{R}$. Then M is isometric to a splitting $M' \times \mathbb{R}$.

Analogous statement generalized to Ricci limits by Cheeger-Colding '97, and generalized to $RCD^*(0, N)$ spaces by Gigli '13:

Splitting Theorem for RCD (Gigli '13). Let $(X, \mathsf{d}, \mathfrak{m})$ be an $RCD^*(0, N)$ space. Assume X contains a line. Then there exists an $RCD^*(0, N-1)$ space $(X', \mathsf{d}', \mathfrak{m}')$ such that $(X, \mathsf{d}, \mathfrak{m})$ is isomorphic as m.m.s. to $(X' \times \mathbb{R}, \mathsf{d}' \otimes \mathsf{d}_E, \mathfrak{m}' \otimes \mathcal{L}^1)$.

Euclidean tangents to $RCD^*(K,N)$ spaces

- Cheeger-Colding '97: for limit spaces the local blow ups are a.e. unique and Euclidean.
- Q: is it true also for $RCD^*(K,N)$ spaces?
- Notation Fixed $\bar{x} \in X$, call $\mathrm{Tan}(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ the set of local blow ups (also called tangent cones) of X at $\bar{x}$.

THM [Gigli-M.-Rajala '13] Let $(X, \mathsf{d}, \mathfrak{m})$ be an $RCD^*(K,N)$ space. Then for $\mathfrak{m}$-a.e. $x \in X$ there exists $n = n(x) \in \mathbb{N}$, $n \leq N$, such that

$$(\mathbb{R}^n, \mathsf{d}_E, \mathcal{L}_n, 0) \in \mathrm{Tan}(X, \mathsf{d}, \mathfrak{m}, x).$$

Idea of proof

1. $\mathfrak{m}$-a.e. $\bar{x} \in X$ is the midpoint of some geodesic
2. Take a sequence of blow ups at such $\bar{x}$, by Gromov compactness and by Stability they converge to a limit $RCD^*(0, N)$ space $(Y, \mathsf{d}_Y, \mathfrak{m}_Y, \bar{y}) \in \mathrm{Tan}(X, \mathsf{d}, \mathfrak{m}, \bar{x})$
3. By the choice of $\bar{x}$, Y contains a line and therefore splits an $\mathbb{R}$ factor, by the splitting thm: $Y \cong Y' \times \mathbb{R}$, Y' $RCD^*(0, N-1)$.
4. Repeating the construction for Y' in place of X we get that there exists a local blow up $\tilde{Y}'$ of Y' that splits an $\mathbb{R}$ factor: $\tilde{Y}' = Y'' \times \mathbb{R}$, Y'' $RCD^*(N-2, 0)$
5. Adapting ideas of Preiss we prove that $\mathfrak{m}$-a.e. tangents of tangents are tangent themselves, i.e. $Y'' \times \mathbb{R}^2 = \tilde{Y}' \times \mathbb{R} \in \mathrm{Tan}(X, \mathsf{d}, \mathfrak{m}, \bar{x})$
6. repeating the scheme iteratively we conclude.

$\square$

Uniqueness of tangents and rectifiability for $RCD^*(K,N)$

Q: In the previous Thm we have existence of a Euclidean tangent cone; but is the tangent cone unique?

THM 2[M.-Naber'14] Let $(X, \mathsf{d}, \mathfrak{m})$ be an $RCD^*(K,N)$ space. Then for $\mathfrak{m}$-a.e. $x \in X$ the tangent cone IS UNIQUE and Euclidean, i.e. there exists $n = n(x) \in \mathbb{N}$, $n \leq N$, such that

$$\{(\mathbb{R}^n, \mathsf{d}_E, \mathcal{L}_n, 0)\} = \mathrm{Tan}(X, \mathsf{d}, \mathfrak{m}, x),$$

More precisely we have
THM 3[M.-Naber'14] [Rectifiability of $RCD^*(K,N)$-spaces] Let $(X, \mathsf{d}, \mathfrak{m})$ be an $RCD^*(K,N)$ space. Then, for every $\varepsilon > 0$ there exists a countable collection $\{R_j^\varepsilon\}_{j\in\mathbb{N}}$ of $\mathfrak{m}$-measurable subsets of X, covering X up to an $\mathfrak{m}$-negligible set, such that each R_j^ε is $1+\varepsilon$-biLipshitz to a measurable subset of $\mathbb{R}^{k_j}$, for some $1 \leq k_j \leq N$, k_j possibly depending on j.

Preliminary remarks

- If X is a Ricci limit space, Thm 2 was first proved by Cheeger-Colding '00: prove hessian estimates on harmonic approximations of distance functions, and use these to force splitting behavior.
- At the time of the work M.-Naber, the notion of a hessian was not at the same level as it is for a smooth manifold, and could not be used in such strength.
- So we proved new estimates: gradient estimates on the excess function and a new almost splitting theorem with excess $\rightsquigarrow$ allows to use the distance functions directly as chart maps. New even in the smooth context.
- In the meantime Gigli and Gigli-Tamanini developed a powerful second order calculus for RCD spaces. Moreover Ambrosio-Honda proved powerful stability properties. Building on top of such more advanced calculus tools, Brué-Semola-Pasqualetto recently gave a proof of the rectifiability of RCD spaces more in the spirit of original Cheeger-Colding proof.

Strategy of proof, 1: the A_k's.

Define

$A_k := \{x \in X \ : \ \exists$ a tangent cone of X at x equal to $\mathbb{R}^k$ but no tangent cone at x splits $\mathbb{R}^{k+1}\}$.

We first prove that

-A_k is $\mathfrak{m}$-measurable (it is difference of analytic sets),

- by THM 1 we get $\mathfrak{m}(X \setminus \bigcup_{k \in \mathbb{N}} A_k) = 0$.

So THM 2-3 are a consequence of the following

THM 4. Let $(X, \mathrm{d}, \mathfrak{m})$ be an $RCD^*(K, N)$-space, and let A_k be as above. Then

(1) For $\mathfrak{m}$-a.e. $x \in A_k$ the tangent cone of X at x is unique and isomorphic to the k-dimensional Euclidean space.

(2) There exists $\bar{\varepsilon} = \bar{\varepsilon}(K, N) > 0$ such that, for every $0 < \varepsilon \leq \bar{\varepsilon}$, A_k is k-rectifiable via $1 + \varepsilon$-biLipschitz maps. More precisely, for each $\varepsilon > 0$ we can cover A_k, up to an $\mathfrak{m}$-negligible subset, by a countable collection of sets U_ε^k with the property that each one is $1 + \varepsilon$-biLipschitz to a subset of $\mathbb{R}^k$.

Strategy of proof, 2: rough idea

1. Given $\bar{x} \in A_k$, for every $0 < \delta << 1$ there exists $r > 0$ such that $\mathsf{d}_{mGH}(B_{\delta^{-1}r}(\bar{x}), (B_{\delta^{-1}r}(0^k)) \leq \delta r$.
2. For some radius $r << R << \delta^{-1} r$ we can then pick points $\{p_i, q_i\}_{i=1,\dots,k} \in X$ corresponding to the bases $\pm Re_i$ of $\mathbb{R}^k$. Define the map
$\vec{d} = \Big(\mathsf{d}(p_1,\cdot) - \mathsf{d}(p_1,\bar{x}), \dots, \mathsf{d}(p_k,\cdot) - \mathsf{d}(p_k,\bar{x})\Big) : B_r(\bar{x}) \to \mathbb{R}^k$.
For δ sufficiently small, $\vec{d}$ is a εr-mGH map $B_r(\bar{x}) \to B_r(0^k)$.
3. MAIN CLAIM: $\exists$ a set $U_\varepsilon \subseteq B_r(\bar{x})$ of almost full measure, i.e. $\mathfrak{m}(B_r(\bar{x}) \setminus U_\varepsilon) \leq \varepsilon$, s.t. $\forall x \in U_\varepsilon$ and $s \leq r$, the restriction map $\vec{d} : B_s(x) \to \mathbb{R}^k$ is an εs-measured Gromov Hausdorff map.
4. From this we can show that the restriction map $\vec{d} : U_\varepsilon \to \mathbb{R}^k$ is in fact $1+\varepsilon$-bilipschitz onto its image. By covering A_k with such sets we will show that A_k is itself rectifiable.

Strategy of proof, 3: two new ingredients

Define $e_{p,q}(y) := \mathsf{d}(p,y) + \mathsf{d}(q,y) - \mathsf{d}(p,q)$, called excess function. In order to get the main claim, two new ingredients

1. Gradient Excess Estimates. We show that the gradient of the excess functions e_{p_i,q_i} of the points $\{p_i, q_i\}$ is small in L^2, more precisely: for the above $\delta > 0$ small enough, then

$$\fint_{B_r(\bar{x})} |De_{p_i,q_i}|^2 \, d\mathfrak{m} \leq \varepsilon_1.$$

2. Almost splitting via excess: given $x \in B_r(\bar{x})$ and $s \in (0,r)$, if $\fint_{B_s(x)} |De_{p_i,q_i}|^2 \, d\mathfrak{m} < \varepsilon_1$, then

$$\mathsf{d}_{mGH}\left(B_s(x), B_s^{\mathbb{R}\times Y}((0,y))\right) < \varepsilon_2 \, s,$$

for some m.m.s. $(Y, \mathsf{d}_Y, \mathfrak{m}_Y, y)$.
I.e.: gradient of excess small in $L^2 \Rightarrow$ close to a splitting.
Proof by contradiction, in the limit we enter into the framework of the arguments of Splitting Theorem.

Strategy of proof, 4: construction of U_ε

Conclusion via a maximal function argument: for $x \in B_r(\bar{x})$ call

$$M(x) := \sup_{s\in(0,r)} \sum_{i=1}^{k} \fint_{B_s(x)} |De_{p_i,q_i}|^2 \, d\mathfrak{m}.$$

Define

$$U_\varepsilon := \{x \in B_r(\bar{x}) : M(x) < \varepsilon\}.$$

By the Gradient Excess Estimates+ $L^1 \to L^{1,weak}$ continuity of maximal function operator
$\Rightarrow$ for $\delta > 0$ small enough we have $\mathfrak{m}(B_r(\bar{x}) \setminus U_\varepsilon) < \varepsilon$.

But $\forall x \in U_\varepsilon$, $\forall s \le r$, by construction,
$\sum_{i=1}^{k} \fint_{B_s(x)} |De_{p_i,q_i}|^2 \, d\mathfrak{m} \le \varepsilon$. An iteration of the almost splitting theorem via excess estimates implies then that

$$\mathrm{d}_{mGH}(B_s(x), B_s(0^k)) \le \varepsilon_2 \, s, \quad \forall s \le r \quad \Rightarrow \quad \text{Main claim.}$$

The measure $\mathfrak{m}$ in the rectifiability

Q: From THM 4 we know that A_k is k-rectifiable. What can we say about $\mathfrak{m} \llcorner A_k$? Is it absolutely continuous wrt to $\mathcal{H}^k$?

THM(Kell-M., De Philippis-Marchese-Rindler, Gigli-Pasqualetto) : YES!

$$\mathfrak{m} \llcorner A_k \ll \mathcal{H}^k$$

Key idea of all the proofs: use result by De Philippis-Rindler (also announced by Csorney-Jones):

"Converse" of Rademacher Theorem:
Let μ be a non-negative Radon measure on $\mathbb{R}^n$ such that every Lipschitz function is differentiable μ-a.e.. Then $\mu \ll \mathcal{L}^n$.

Constancy of the dimension

- Combining the above results, we have that if $(X, \mathsf{d}, \mathfrak{m})$ is an $RCD^*(K,N)$ space, then $X = \cup_{k\in\mathbb{N},k=1}^{[N]} A_k$. Each A_k is k-rectifiable and $\mathfrak{m}\llcorner A_k \ll \mathcal{H}^k$.
  For Ricci-limits, already established by Cheeger-Colding '00.
- Q: Is it possible to have more than one A_k with $\mathfrak{m}(A_k) > 0$? In other terms: is the dimension of the Euclidean tangent spaces constant $\mathfrak{m}$-a.e. or it is possible to have non-negligible strata of different dimension?
- For Ricci-limits answered by Colding-Naber '12: along a geodesic $(\gamma_t)_{t\in[0,1]}$ the tangent cones are Hölder-continuous in $t \in [\epsilon, 1-\epsilon]$ wrt GH topology
  $\rightsquigarrow$ the dimension of tangent cones cannot jump of dimension along a geodesic, leading to:
- THM (Colding-Naber '12): If $(X, \mathsf{d}, \mathfrak{m})$ is a Ricci limit, then there exists $k \in \mathbb{N}$ such that $\mathfrak{m}(X \setminus A_k) = 0$.

Constancy of the dimension for $RCD^*(K, N)$

THM (Brué-Semola '18): If $(X, \mathrm{d}, \mathfrak{m})$ is an $RCD^*(K, N)$ space, then there exists $k \in \mathbb{N} \cap [1, [N]]$ such that $\mathfrak{m}(X \setminus A_k) = 0$.

Similarities and differences with Colding-Naber

- CN prove estimates on smooth approximation and pass them into the limit. For RCD there is no smooth approximating sequence, so need to work directly on the non-smooth space.
- While CN look at how the geometry varies along a minimizing geodesic, BS look at how the geometry varies along a Wasserstein geodesic.
- BS avoid using second order differentiation formula (which was key in CN) and obtain quantitative estimates on flows which is new even in the smooth setting

Heuristic of Brué-Semola's approach

- Fact: Let M be a smooth connected differentiable manifold. Then, for any $x, y \in M$, there exists a smooth diffeomorphism $\varphi : M \to M$ such that $\varphi(x) = y$.
- Strategy: Build such φ as a flow map at a suitable time of a suitable vector field.

Output

- In smooth setting: this approach gives new quantitative estimates (in the same spirit of Colding-Naber)
- In non-smooth setting: new results also at the qualitative level (constancy of the dimension of $RCD^*(K, N)$ spaces).

Rough strategy of Brué-Semola's approach

1. Find a "rich enough" class of vector fields admitting a "regular enough" flow
   - vector fields $\leftrightarrow$ derivations in sense of Waever '00 (see also Gigli '14)
   - flow of Sobolev vector fields $\leftrightarrow$ Regular Lagrangian flow in the sense of Di Perna-Lyons '89 Ambrosio '04 and Ambrosio-Trevisan '14.
2. Investigate the regularity of such flows.
   - BS establish a very powerful Lusin-Lipschitz regularity (inspired by previous work of Crippa-DeLellis '08): basically, the flow is Lipschitz out of a set of small measure.
   - Key idea (inspired by previous work by Colding): prove the estimates in terms of $d_G := 1/G$, where G is the Green function of the Laplacian instead that in terms of the distance function.
3. Prove rigidity statements for Lusin-Lipschitz maps (roughly, they cannot map a piece of $\mathbb{R}^{k_1}$ to $\mathbb{R}^{k_2}$ with $k_1 \neq k_2$).
4. Combine the ingredients and prove the constancy of the dimension.

Recent news about structure

- Hölder continuity of tangent cones along interior of geodesics in $RCD^*(K,N)$ spaces recently established by Qin Deng '20. Difference with Colding-Naber:
  - CN perform an iteration on tubular neighbourhoods of a geodesic iterating on the radius of the tubular neighbourhood.
  - Deng does an iteration in the "t" parameter of the geodesic γ_t.
- Wei-Pan '21 recently gave examples of Ricci limit spaces with non-integer Hausdorff dimension and where the Hausdorff dimension of the singular set exceeds that of the regular set.

N-dimensional $RCD^*(K, N)$ spaces, $N \in \mathbb{N}$

- Analogy with non-collapsed Ricci limit spaces of Cheeger-Colding '97
- Honda '18: $(X, \mathrm{d}, \mathfrak{m})$ is a (compact) $RCD^*(K, N)$ space of Hausdorff dimension $N \in \mathbb{N}$, then $\mathfrak{m} = const \times \mathcal{H}^N$.
- If $(X, \mathrm{d}, \mathcal{H}^N)$ is $RCD^*(K, N)$, then Bishop inequality holds (Kitabeppu '17). Case $K = 0$ (for general K compare with suitable model spaces): $\mathcal{H}^N(B_r(x)) \leq \omega_N r^n$
- De Philippis-Gigli '18:
  - If $(X, \mathrm{d}, \mathcal{H}^N)$ is $RCD^*(K, N)$, then one has a stratification of the singular set as for non-collapsed Ricci limits (Ch-Co '97)
  - Colding's volume convergence holds: For $K \in \mathbb{R}$, $N \geq 2$, $R \in (0, \infty)$ denote $\mathbb{B}(K, N, R)$ the collection of all (equivalence classes up to isometry of) closed balls of radius R in $RCD^*(K, N)$ spaces equipped with the Gromov-Hausdorff distance. Then the map $\mathbb{B}(K, N, R) \ni Z \mapsto \mathcal{H}^N(Z)$ is real valued and continuous.
- Antonelli-Brué-Semola '19: quantitative stratification of the singular set as for non-collapsed Ricci lim. (Cheeger-Naber'13)

N-dimensional $RCD^*(K,N)$ spaces, $N \in \mathbb{N}$

THM (Kapovitch-M. '19, after Cheeger-Colding '97 for Ricci limits) Let $(X, \mathrm{d}, \mathcal{H}^N)$ be an N-dim $RCD^*(K,N)$. Then

- there exists a dense open subset $M \subset X$ with $\mathcal{H}^N(X \setminus M) = 0$, bi-Hölder homeomorphic to a smooth N-dim manifold;
- the (possibly empty) boundary part $\partial X \subset X$ (defined inductively using tangent cones) has Hausdorff dim $\leq N-1$;
- $X \setminus \big(\partial X \cup M\big)$ has Hausdorff dimension at most $N-2$.

$\rightsquigarrow$ X is the disjoint union of a manifold part of dimension N, a boundary part of Hausdorff dimension at most $N-1$ and a singular set of Hausdorff dim $\leq N-2$.

THM (Brué-Naber-Semola '20) The boundary ∂X is $N-1$ rectifiable and homemorphic to a smooth $N-1$-dim manifold away from a set of Hausdorff dim $\leq N-2$.

BNS's Thm is new even for Ricci limit spaces!

SUMMARY OF $RCD^*(K, N)$

Good properties of $RCD^*(K, N)$

- Stability under pmGH convergence (Ambrosio-Gigli-Savaré and Gigli-M.-Savaré)
- Equivalent to contractivity (EVI) of heat flow in W_2 in case $N = \infty$ (Ambrosio-Gigli-Savaré, Ambrosio-Gigli-M.-Rajala)
- Equivalent to Bochner inequality (for $N = \infty$ Ambrosio-Gigli-Savaré, for $N \in [1, \infty)$ Erbar-Kuwada-Sturm Vs Ambrosio-M.-Savaré)
- Implies Li-Yau inequalities (Garofalo-M. and Jiang)
- Implies Cheeger-Gromoll Splitting Theorem (Gigli)
- Local structure: Euclidean tangent cones (Gigli-M.-Rajala and M.-Naber), rectifiability (M.-Naber), a.e. unique dimension of tangent cones (Brué-Semola)
- Implies that Isometries are a Lie Group (Guijarro-Rodriguez, Sosa)
- Implies existence of a universal cover + classical Theorems on the (revised) fundamental group (M.-Wei)
- Local to Global (Ambrosio-M.-Savaré, Cavalletti-Milman)

Examples of *RCD*-spaces

- Ricci limits, no matter if collapsed or not and no matter if the dimension is bounded above or not (in the first case get $RCD^*(K,N)$, in the latter get $RCD^*(K,\infty)$)
- Finite dimensional Alexandrov spaces with curvature bounded below (Perelman 90'ies and Otsu-Shioya '94: *Ch* is quadratic, Petrunin '12: CD is satisfied)
- Weighted Riemannian manifolds with Bakry-Émery $N - Ricci \geq K$: i.e. (M^n, g) Riemannian manifold, let $\mathfrak{m} := \Psi\, vol_g$ for some smooth function $\Psi \geq 0$, then $Ric_{g,\Psi,N} := Ric_g - (N-n)\frac{\nabla^2 \Psi^{1/N-n}}{\Psi^{1/N-n}} \geq Kg$ iff $(M, \mathrm{d}_g, \mathfrak{m})$ is $RCD^*(K,N)$.
- Cones or spherical suspensions over $RCD^*(N-1,N)$spaces (Ketterer)
- Quotients, orbifolds, metric-measure foliations with Ricci bounded below (GalazGarcia-Kell-M.-Sosa).
- Stratified spaces with Ricci bounded below and cone angle$\leq 2\pi$ (Bertrand-Ketterer-Mondello-Richard).

Conclusion

In doing Riemannian geometry one naturally encounters non smooth spaces

- when taking limits of Riemannian manifolds (procedure used often, e.g. contradiction arguments, blow up arguments, singularities in geometric flows),
- when taking quotients, cones, foliations of Riemannian manifolds.

If the smooth spaces we started with have Ricci bounded below, then the non smooth spaces arising are RCD.

$\rightarrow$ $RCD^*(K, N)$ spaces can be seen as an extension of the class of smooth Riemannian manifolds with Ricci $\geq K$, which is closed under many natural geometric and analytic operations.
Next lecture we will see some smooth appications.