Metric measure spaces satisfying Ricci curvature lower bounds
Lecture 4

Andrea Mondino (University of Oxford)

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Recap

- Lecture 1: $CD(K,N)$ spaces, $K \in \mathbb{R}$, $N \in [1,+\infty]$.
  Motivation, definition, stability.
- Lecture 2: $RCD\&RCD^*(K,N)$ spaces, $K \in \mathbb{R}$, $N \in [1,+\infty]$.
  Motivation, definition, stability, Bochner inequality.
- Lecture 3: structure theory of $RCD^*(K,N)$ spaces,
  $K \in \mathbb{R}$, $N \in [1,+\infty)$.
- Lecture 4: sharp & rigid geometric & functional inequalities

Smooth applications of synthetic techniques

Plan of the talk

GOAL: discuss some recent geometric inequalities in their quantitative form for smooth Riemannian manifolds with OT transport tools, more precisely using techniques coming from synthetic Ricci curvature lower bounds

- Quantitative Lévy-Gromov isoperimetric inequality. (with F. Cavalletti and F. Maggi)
- Almost Euclidean isoperimetric inequality in a small ball in a manifold with Ricci curvature bounded below, motivated by Ricci flow (with F. Cavalletti)
- Quantitative Obata's rigidity Theorem. (with F. Cavalletti and D. Semola)

Isoperimetric problem

One of oldest problems in mathematics, roots in myths of 2000 years ago (Queen Dido's problem).
Q: Given a space X and a volume v, what is the minimal amount of (boundary) area needed to enclose the volume $v > 0$?

Examples

- $X = \mathbb{R}^n \rightsquigarrow$ Euclidean isoperimetric inequality:
  For all $E \subset \mathbb{R}^n$ it holds $|\partial E| \geq |\partial B|$ where B is a round ball s.t. $|B| = |E|$.
- $X = \mathbb{S}^n$ analogous:
  For all $E \subset S^n$ it holds $|\partial E| \geq |\partial B|$ where B is a metric ball (i.e. a spherical cap) s.t. $|B| = |E|$

RK: In both of the examples the space is fixed (Euclidean space of Sphere), such a space contains a model subset (metric ball), and any subset of the space is compared with such a model subset.

Lévy-Gromov inequality

Besides the Euclidean one, probably the most famous isoperimetric inequality is the Lévy-Gromov isoperimetric inequality:

Lévy-Gromov isoperimetric inequality

Let (M^n, g) be a Riemannian manifold with $Ric_g \geq (n-1)g$ and $E \subset M$ domain with smooth boundary ∂E.

Let $\mathbb{S}^n$ be the round sphere of unit radius (so that $Ric \equiv n-1$), and $B \subset \mathbb{S}^n$ be a metric ball s.t. $\frac{|E|}{|M|} = \frac{|B|}{|\mathbb{S}^n|}$. Then

$$\frac{|\partial E|}{|M|} \geq \frac{|\partial B|}{|\mathbb{S}^n|}$$

RK. (1) In the (LGI) the space is NOT fixed: any subset in any manifold with $Ric \geq n-1$ is compared with the model subset (i.e. spherical cap) in the model space (i.e. the sphere).

(2) (LGI) is global in the space, i.e. it does not depend just on E but also on $M \setminus E$: if one changes the space locally outside of E, the lhs in (LGI) may change since $|M|$ may change.

Equivalent way to state LG inequality in terms of isoperimetric profile

- Given a Riemannian manifold (M, g), define its *isoperimetric profile function* as

$$\mathcal{I}_{(M,g)}(v) := \inf \left\{ \frac{|\partial E|}{|M|} \, : \, \frac{|E|}{|M|} = v \right\}, \ \forall v \in [0,1].$$

- Lévy-Gromov inequality can be stated as: Given (M^n, g) with $Ric_g \geq (n-1)g$ then

$$\mathcal{I}_{(M,g)}(v) \geq \mathcal{I}_{\mathbb{S}^n}(v), \ \forall v \in [0,1].$$

Rigidity and almost rigidity in the Lévy-Gromov inequality

- Rigidity: If there exists $E \subset M$ with $\frac{|E|}{|M|} = v \in (0,1)$ satisfying $\frac{|\partial E|}{|M|} = \mathcal{I}_{(M,g)}(v) = \mathcal{I}_{\mathbb{S}^n}(v)$, then
  1) $(M^n, g) \simeq \mathbb{S}^n$ isometric
  2) $E \simeq B$ metric ball.
- Question: Stability? i.e. If "=" in (LGI) is almost attained,
  Q1) What can we say on (M^n, g)? Is it close to a sphere? In which sense?
  Q2) What can we say on E? Is it close to a metric ball? In which sense?

About Question 1

THM 1 (Particular case of Bérard-Besson-Gallot, Inv. Math. 1985)
Given (M^n, g) with $Ric_g \geq (n-1)g$ and $\mathrm{diam}(M) = D$ (recall from Bonnet-Myers $D \in (0, \pi)$) then

$$\frac{\mathcal{I}_{(M,g)}(v)}{\mathcal{I}_{\mathbb{S}^n}(v)} \geq \left(\frac{\int_0^{\pi/2} (\cos t)^{n-1} dt}{\int_0^{D/2} (\cos t)^{n-1} dt} \right)^{1/n}$$

RK: 1) E. Milman improves THM 1 to a sharp version, JEMS 2015.
2) rhs is ≥ 1 so the result sharpens the classical LGI
3) It follows that there exists $C_{n,v} > 0$ such that if for some $v \in (0,1)$ it holds $\mathcal{I}_{(M,g)}(v) \leq \mathcal{I}_{\mathbb{S}^n}(v) + \delta$, then

$$\pi - D \leq C_{n,v} \delta^{1/n}.$$

$\rightsquigarrow d_{GH}(M, S(X)) \leq \varepsilon(\delta)$ by Cheeger-Colding Almost Maximal Diameter Thm (Annals of Math. 1996)

Answering Question 2 in Euclidean setting

Quantitative Euclidean isoperimetric inequality
(Fusco-Maggi-Pratelli, Annals of Math. 2008)
There exists $C_n > 0$ such that for every $E \subset \mathbb{R}^n$ there exists a round ball $B \subset \mathbb{R}^n$ with $|E| = |B|$ and

$$\frac{|E \Delta B|}{|E|} \leq C_n \left(\frac{|\partial E|}{|\partial B|} - 1 \right)^{1/2}$$

RK: 1) the rhs is the so-called "isoperimetric deficit" and is zero iff E is a ball (by rigidity in EII).
2) The proof of FMP is via a "quantitative symmetrization".
3) Alternative proof of the result via Brenier L^2-Optimal Transport map (by Figalli-Maggi-Pratelli, Inv. Math. 2010) and via regularity theory and selection principle (Cicalese-Leonardi, ARMA 2012).

Answering Question 2 in spherical setting

Quantitative spherical isoperimetric inequality
(Bogelein-Duzaar-Fusco, Adv. Calc. Var. 2015)
For every $v \in (0,1)$ and every $n \geq 2$ there exists $C_{n,v} > 0$ with the following property.
For every $E \subset \mathbb{S}^n$ with $\frac{|E|}{|\mathbb{S}^n|} = v$ there exists a metric ball $B \subset \mathbb{S}^n$ with $|B| = |E|$ such that

$$|E \Delta B| \leq C_{n,v} \left(\frac{|\partial E|}{|\mathbb{S}^n|} - \mathcal{I}_{\mathbb{S}^n}(v) \right)^{1/2}$$

Proof: along the same lines of Cicalese-Leonardi's selection principle.

Difficulties about Question 2: quantitative Lévy-Gromov inequality

The above quantitative isoperimetric inequalities are for a fixed space ($\mathbb{R}^n$ or $\mathbb{S}^n$), with the highest possible degree of symmetry.
LGI is for any (M^n, g) with $Ric_g \geq (n-1)g$
$\rightsquigarrow$ No fixed space and no symmetry.
$\rightsquigarrow$ The above approaches seem not to be applicable:

- Symmetrization (FMP): since M is not symmetric it makes little sense to speak of symmetrization (even if our approach via localization can be seen as a sort of symmetrization).
- Brenier Map, L^2-OT (FMP): works in $\mathbb{R}^n$ but already in $\mathbb{S}^n$ it is an open problem to prove spherical isoperimetric inequality via Brenier Map.
- Selection Principle (CL): would need smooth convergence of metrics while here the natural convergence is mGH.

$\rightsquigarrow$ our approach: localization via L^1-Optimal Transport.

Brief history of localization

The localization technique is a way to reduce an a-priori complicated high dimensional problem to a family of (hopefully) simpler 1-dimensional problems.

- In $\mathbb{R}^n$ or $\mathbb{S}^n$, using the high symmetry of the space, 1-D localizations can be usually obtained via iterative bisections
  - Roots in a paper by Payne-Weinberger '60 about sharp estimate of 1^{st} eigenvalue of Neumann Laplacian in compact convex sets of $\mathbb{R}^n$
  - Formalized by Gromov-V. Milman '87, Kannan - Lovász - Simonovits '95
- Extended by B. Klartag '14 to Riemannian manifolds via L^1-optimal trasport: no symmetry but still heavily using the smoothness of the space (estimates on 2^{nd} fundamental form of level sets of the Kantorovich potential φ)
- Extension to non-smooth spaces (more precisely: essentially non-branching $CD^*(K, N)$ spaces) by Cavalletti-M. '15.

The result: quantitative Lévy-Gromov inequality

THM 2 (Cavalletti-Maggi-M. CPAM '19)
For every $v \in (0,1)$ and $n \geq 2$ there exists $C_{n,v} > 0$ with the following properties.
Let (M^n, g) be with $Ric_g \geq (n-1)g$. For every $E \subset M$ with $\frac{|E|}{|M|} = v$ there exists a metric ball $B \subset M$ with $|B| = |E|$ such that

$$|E \Delta B| \leq C_{n,v} \left(\frac{|\partial E|}{|M|} - \mathcal{I}_{\mathbb{S}^n}(v) \right)^{\frac{n}{n^2+n-1}}$$

In particular, if $E \subset M$ is an isoperimetric subset with $\frac{|E|}{|M|} = v$, then

$$|E \Delta B| \leq C_{n,v} \left(\mathcal{I}_{(M,g)}(v) - \mathcal{I}_{\mathbb{S}^n}(v) \right)^{\frac{n}{n^2+n-1}}$$

RK Difference with (QEII) or (QSII): here $E \subset M$ and $|\partial E|$ is compared with $\mathcal{I}_{\mathbb{S}^n}$ (not of $\mathcal{I}_{(M,g)}$) via a "Lévy-Gromov isoperimetric deficit".

The result holds in higher generality

Actually we prove THM1 and THM 2 more generally for essentially non-branching $CD^*(N-1,N)$ metric measure spaces spaces.
These are (a priori) non-smooth spaces of dimension $\leq N$ and Ricci $\geq N-1$ in a synthetic sense via OT (Lott-Sturm-Villani).
Examples entering this class of spaces:

- Weighted manifolds with N-Bakry-Émery Ricci tensor bounded below by $N-1$
- Measured Gromov Hausdorff limits of Riemannian N-dimensional manifolds satisfying $Ric_g \geq (N-1)g$ and more generally the class of $RCD^*(N-1,N)$ spaces.
- Finite dimensional Alexandrov spaces with curvature ≥ 1
- Finsler manifolds satisfying $CD^*(N-1,N)$

Part 2. Ricci flow, Perelman's Pseudo Locality Theorem and Almost Euclidean isoperimetric inequalities.

Perelman's Pseudo-locality Theorem

THM[Theorem 10.1, Perelman's first Ricci flow paper 2002] For every $\alpha > 0$ there exists $\delta > 0, \varepsilon > 0$ with the following property. Suppose we have a smooth complete solution to the Ricci flow $(g_{ij})_t = -2R_{ij}$, $0 \le t \le \varepsilon^2$, and assume that at $t = 0$ we have

$$R_{g_0}(x) \ge -1 \quad \& \quad |\partial\Omega|_{g_0} \ge (1-\delta)\, c_n |\Omega|_{g_0}^{(n-1)/n}, \ \forall x, \Omega \subset B_1(x_0),$$

where c_n is the Euclidean isoperimetric constant.
Then we have an estimate $|Rm|(x,t) \le \alpha t^{-1} + \varepsilon^{-2}$ whenever $0 < t \le \varepsilon^2$, $\mathrm{d}_{g_t}(x, x_0) < \varepsilon$.

RK: The non-linearity of Ricci flow here helps: if we have good geometric control on ball, and no assumptions outside (apart from completeness), the Ricci flow for small times improves the geometric control in the ball regardless how bad the manifold is outside.

Perelman's Pseudo-locality Theorem revisited by Tian&Wang

THM[Tian-Wang JAMS 2015] For every $\alpha > 0$ there exists $\delta > 0, \varepsilon > 0$ with the following property. Suppose we have a smooth solution to the Ricci flow $(g_{ij})_t = -2R_{ij}$, $0 \leq t \leq \varepsilon^2$, and assume that at $t = 0$ we have

$$Ric_{g_0}(x) \geq -\delta^2 g_0 \text{ on } B_1(x_0) \quad \& \quad |B_1(x_0)|_{g_0} \geq (1-\delta)\,\omega_n.$$

Then $|Rm|(x,t) \leq \alpha t^{-1} + \varepsilon^{-2}$ for $0 < t \leq \varepsilon^2$, $d_{g_t}(x, x_0) < \varepsilon$.

RK: - From Bishop Gromov we have $|B|_{g_0} \leq (1 + C\,\delta)\,\omega_n$, so the condition $|B|_{g_0} \geq (1-\delta)\,\omega_n$ is an almost maximal volume assumption.
-The proof by Tian-Wang is highly technical and not at all a straightforward corollary of Perelman's Pseudo-locality Theorem.

Almost Euclidean isoperimetric inequality

Q: do the assumptions of Tian-Wang's Pseudo-locality imply the assumptions of Perelman's Pseudo-locality? I.E.

$$Ric_{g_0}(x) \geq -\delta^2 g_0 \;\&\; |B|_{g_0} \geq (1-\delta)\,\omega_n \overset{?}{\Rightarrow} |\partial\Omega|^n_{g_0} \geq (1-\varepsilon)\,c_n |\Omega|^{n-1}_{g_0}$$

for all $\Omega \subset B_\varepsilon(x_0)$.

THM[Cavalletti-M. IMRN '18] For every $N \in [2,\infty) \cap \mathbb{N}$ there exist $\bar{\varepsilon}_N, \bar{\delta}_N, C_N > 0$ such that the next holds.
Let (M,g) be a smooth N-dim. Riem. manifold and let $\bar{x} \in M$. Assume that $B_1(\bar{x})$ is rel. compact and for some $\delta \in [0, \bar{\delta}_N]$

$$Ric_g \geq -\delta^2 g \;\text{ on } B_1(\bar{x}) \quad \& \quad |B_1(\bar{x})| \geq (1-\delta)\omega_N$$

Then for every subset $E \subset B_{\varepsilon_N}(\bar{x})$:

$$|\partial E|_g \geq N\omega_N^{1/N}(1 - C_N\delta)\,|E|_g^{\frac{N-1}{N}}.$$

RK Actually we prove the corresponding statement more generally for a m. m. space $(X, \mathsf{d}, \mathfrak{m})$ which is essentially non-branching, $CD_{loc}(-\delta, N)$ on a ball $B_1(\bar{x})$ and $\mathfrak{m}(B_1(\bar{x})) \geq (1-\delta)\omega_N$.

Combining Colding's volume convergence Theorem (Annals of Math. '97) with the above result we get:

COR[Cavalletti-M. IMRN '18] For every $N \in [2, \infty) \cap \mathbb{N}$ there exist $\bar{\varepsilon}_N, \bar{\delta}_N, C_N > 0$ such that the next holds.
Let (M, g) be a smooth N-dim. Riem. manifold and let $\bar{x} \in M$. Assume that $B_1(\bar{x})$ is rel. compact and for some $\delta \in [0, \bar{\delta}_N]$, it holds:

$$Ric_g \geq -\delta^2 g \ \text{ on } B_1(\bar{x}) \quad \& \quad \mathsf{d}_{GH}(B_1(\bar{x}), B_1^{\mathbb{R}^N}) \leq \delta.$$

Then for every subset $E \subset B_{\varepsilon_N}(\bar{x})$:

$$|\partial E|_g \geq N\omega_N^{1/N}(1 - C_N\delta)\,|E|_g^{\frac{N-1}{N}}.$$

RK closeness in GH-distance is a priori a very weak assumption (a manifold is δ-GH close to a δ-net which is a discrete space); so it is remarkable that GH-close + lower Ricci bound $\Rightarrow$ almost Euclidean isoperimetric inequality.

Some comments, 1.

Q: Why the almost Euclidean isoperimetric inequality was an open problem?

Classical method for proving Lévy-Gromov isoperimetric inequality in a nutshell:

1. in a compact manifold, for every fixed volume v there is a minimizer Ω of the perimeter having volume v.
2. $\partial\Omega$ is smooth (up to a singular set of large codimension) and the smooth part has constant mean curvature (the regularity is now classical but it is not trivial at all!).
3. Using the regularity of $\partial\Omega$ (crucial: regular part has CMC) perform computations $\rightsquigarrow$ get a lower bound on $|\partial\Omega|$ (so a fortiori get a lower bound of the perimeter of any set since Ω is a minimizer).

Some comments, 2.

DIFFICULTY If we want to prove an AE isoperimetric ineq on $B_1(\bar{x})$

- A minimizing sequence for the perimeter can approach $\partial B_1(\bar{x})$ and so the minimizer Ω will hit $\partial B_1(\bar{x})$.
- On the contact region we have an obstacle problem, regularity is more tricky (partial regularity by Caffarelli in 70ies); in any case $\partial\Omega \cap \partial B_1(\bar{x})$ may not have constant mean curvature (if $\partial B_1(\bar{x})$ has not)
- $\rightsquigarrow$ not in good shape to perform computations of Lévy-Gromov on the minimizer.

Some comments, 3.

- What we do: Via 1-D localization, we prove the lower bound on the perimeter of EVERY subset, not just of the minimizers, without any regularity assumption.
- $\rightsquigarrow$ One uses synthetic Ricci curvature lower bounds via optimal transport to prove a new smooth statement.

PART 3:
A QUANTITATIVE OBATA THEOREM

Spectral gap

Sharp Lichnerowicz spectral gap: Let (M, g) be n-dim with Ric$\geq n-1$ and let $f \in Lip(M)$ with $\int_M f \, dvol_g = 0$ then

$$\int_M f^2 \, dvol_g \leq \frac{1}{n} \int_M |\nabla f|^2 \, dvol_g.$$

- Given (M, g), the first non-zero eigenvalue of the Neumann Laplacian is:

$$\lambda_1(M) := \inf \left\{ \int_M |\nabla f|^2 dvol_g \, : \, \|f\|_{L^2(M)} = 1, \int_M f \, dvol_g = 0 \right\}$$

- Lichnerowicz inequality can be stated as: let (M, g) be n-dim with Ric$\geq n-1$, then

$$\lambda_1(M) \geq n = \lambda_1(\mathbb{S}^n)$$

Rigidity and Stability of Lichnerowicz inequality

- Rigidity: Obata's Theorem 1962
  Let (M, g) be n-dim with Ric$\geq n-1$.
  Then $\lambda_1(M) = n$ iff (M, g) is isometric to $\mathbb{S}^n$.
  Note: First eigenfunction on $\mathbb{S}^n$ is $\cos(\mathrm{d}_x)/c_n$ for any $x \in \mathbb{S}^n$.

- Stability? i.e. if "$=$" in spectral gap is almost attained:
  - Cheng '75, Croke '82: $\lambda_1(M) \simeq n$ iff $diam(M) \simeq \pi$
  - Bérard-Besson-Gallot '85: $\lambda_1(M) - n \geq C_n(\pi - diam(M))^n$
  - Bertrand '07: stability of eigenfunctions: there exists a function $\tau(t) \to 0$ as $t \to 0$ s.t.
    if $\lambda_1(M) \leq n + \epsilon$, then $\|f - cos(\mathrm{d}_x)\|_\infty \leq \tau(\epsilon)$ for f first eigenfunction.

- Question: can we make quantitative Bertrand's result (i.e. quantify τ) and generalize it to a function with almost optimal Rellich quotient (but non-necessarily eigenfunction)?
  i.e. if $f \in Lip(M)$, $\|f\|_2 = 1$, $\int_M |\nabla f|^2 \, dvol_g \simeq n$
  is it true that $f \simeq cos(\mathrm{d}_x)$ for some $x \in M$?

The result: Quantitative Obata's Theorem

THM(Cavalletti-M.-Semola. '19)
For every $n \geq 2$ there exists $C_n > 0$ with the following properties. Let (M, g) be n-dim with Ric$\geq n-1$. For every $f \in Lip(M)$ with

$$\int_M f \, dvol_g = 0, \quad \int_M f^2 \, dvol_g = 1,$$

there exists a point $x \in M$ such that

$$\|f - \cos(\mathrm{d}_x)/c_n\|_2 \leq C_n \left(\int_M |\nabla f|^2 \, dvol_g - n \right)^{\frac{1}{8n+4}} .$$

In particular, if f is a first eigenfunction, then

$$\|f - \cos(\mathrm{d}_x)/c_n\|_2 \leq C_n \left(\lambda_1(M) - \lambda_1(\mathbb{S}^n) \right)^{\frac{1}{8n+4}} .$$

RK: Proved more generally for essentially non branching $CD^*(N-1, N)$ spaces.

PART 4:
SOME IDEAS OF THE PROOFS

Technique: 1-D localization

Let $(X, \mathsf{d}, \mathfrak{m})$ be e.n.b. $CD^*(K,N)$, with $\mathfrak{m}(X) = 1$. Given $E \subset X$ we can find a "1-D localization" $\{X_\alpha\}_{\alpha \in Q}$ of X, i.e.

1. $\{X_\alpha\}_{\alpha \in Q}$ is (essentially) a partition of X, i.e. $\mathfrak{m}(X \setminus \mathring{\bigcup}_{\alpha \in Q} X_\alpha) = 0$
2. Disintegration of $\mathfrak{m}$ wrt $\{X_\alpha\}_{\alpha \in Q}$ (kind of non-straight Fubini): $\mathfrak{m} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha)$, with $\mathfrak{q}(Q) = 1$ and $\mathfrak{m}_\alpha(X_\alpha) = \mathfrak{m}_\alpha(X) = 1$ for $\mathfrak{q}$-a.e. $\alpha \in Q$
3. X_α is a geodesic in X and $(X_\alpha, |\cdot|, \mathfrak{m}_\alpha)$ is a $CD(K,N)$ space
4. $\mathfrak{m}_\alpha(E \cap X_\alpha) = \mathfrak{m}(E)$, for $\mathfrak{q}$-a.e. $\alpha \in Q$

How to obtain a localization: Consider the OT-problem with $c(x,y) = \mathsf{d}(x,y)$ between

$$\mu_0 := (\chi_E / \mathfrak{m}(E))\mathfrak{m} \quad \text{and} \quad \mu_1 := (\chi_{X \setminus E} / \mathfrak{m}(X \setminus E))\mathfrak{m}.$$

X_α will be integral curve of $-\nabla \varphi$, with φ Kantorovich potential.

More on how to construct a 1-D localization

- Recall that $\mathfrak{m}(X) = 1$, fix $E \subset X$ with $\mathfrak{m}(E) \in (0,1)$,
- Let $\mu_0 := \frac{\chi_E}{\mathfrak{m}(E)}\,\mathfrak{m}$ and $\mu_1 := \frac{1-\chi_E}{1-\mathfrak{m}(E)}\,\mathfrak{m} = \frac{\chi_{X\setminus E}}{\mathfrak{m}(X\setminus E)}\,\mathfrak{m}$
- Consider the L^1-optimal transport problem

$$\inf_\gamma \left\{ \int_{X\times X} \mathsf{d}(x,y)\,d\gamma \;:\; \gamma \in \mathcal{P}(X\times X), (\pi_1)_\sharp\gamma = \mu_0, (\pi_2)_\sharp\gamma = \mu_1 \right\}$$

- By Optimal Transport techniques there exists a minimizer $\gamma \in \mathcal{P}(X\times X)$ and a 1-Lipschitz function $\varphi : X \to \mathbb{R}$ called Kantorovich potential such that, denoted

$$\Gamma := \{(x,y) \in X\times X \;:\; \varphi(x) - \varphi(y) = \mathsf{d}(x,y)\},$$

  γ is concentrated on Γ.
- The relation $\sim$ on X given by $x \sim y$ iff $(x,y)\in\Gamma$ or $(y,x)\in\Gamma$ is an equivalence relation on X (up to an $\mathfrak{m}$-negligible subset) and the equivalence classes are geodesics. $\rightsquigarrow$ partition of X into geodesics driven by E
- More work to prove properties 3. and 4.

Why L^1-trasport?

- It is more standard to consider the L^2-optimal transport problem: given $\mu_0, \mu_1 \in \mathcal{P}(X)$ let
$$\inf_{\gamma} \left\{ \int_{X\times X} \mathsf{d}(x,y)^2 \, d\gamma \ : \ \gamma \in \mathcal{P}(X\times X), (\pi_1)_\sharp \gamma = \mu_0, (\pi_2)_\sharp \gamma = \mu_1 \right\}$$
Which defines a metric W_2 on $\mathcal{P}(X)$.
- If $(\mu_t)_{t\in[0,1]}$ is a W_2-geod from μ_0 to μ_1, then μ_t concentrates on t-intermediate points of geodesics from $supp(\mu_0)$ to $supp(\mu_1)$:
$\mu_t(\{\gamma(t) \ : \gamma \, geod, \gamma(0) \in \mathrm{supp}(\mu_0), \gamma(1) \in \mathrm{supp}(\mu_1)\}) = 1,$
- moreover, from d^2-monotonicity, if γ_1 and γ_2 are such geodesics with $\gamma_1(0) \neq \gamma_2(0)$ then $\gamma_1(t) \neq \gamma_2(t)$ in a.e. sense.
$\rightsquigarrow$ the L^2-transport at time t is given by an ess. inj. map.
- BUT it may happen $\gamma_1(s) = \gamma_2(t)$ for $s \neq t$
$\rightsquigarrow$ L^2-transport does not induce an equivalence relation.
- On the other hand L^1 transport does induce an equivalence relation into rays where the transport is performed
$\rightsquigarrow$ partition of the space into $1D$ objects.

Lévy-Gromov inequality via Localization

Let $(X, \mathsf{d}, \mathfrak{m})$ be an e.n.b. $CD^*(N-1, N)$ space.
Assume that for $E \subset X$ we can find a 1-D localization as above then

$$
\begin{aligned}
\mathfrak{m}^+(E) \quad &:= \quad \liminf_{\varepsilon \to 0^+} \frac{\mathfrak{m}(E^\varepsilon) - \mathfrak{m}(E)}{\varepsilon} \\
&= \quad \liminf_{\varepsilon \to 0^+} \int_Q \frac{\mathfrak{m}_\alpha(E^\varepsilon) - \mathfrak{m}_\alpha(E)}{\varepsilon} \, \mathfrak{q}(d\alpha) \quad \text{by } 2. \\
&\geq \quad \int_Q \liminf_{\varepsilon \to 0^+} \frac{\mathfrak{m}_\alpha(E^\varepsilon \cap X_\alpha) - \mathfrak{m}_\alpha(E \cap X_\alpha)}{\varepsilon} \, \mathfrak{q}(d\alpha) \quad \text{by } 2. \\
&\geq \quad \int_Q \liminf_{\varepsilon \to 0^+} \frac{\mathfrak{m}_\alpha((E \cap X_\alpha)^\varepsilon \cap X_\alpha) - \mathfrak{m}_\alpha(E \cap X_\alpha)}{\varepsilon} \, \mathfrak{q}(d\alpha), \\
&\qquad\quad \text{by } E^\varepsilon \cap X_\alpha \supset (E \cap X_\alpha)^\varepsilon \cap X_\alpha \\
&\geq \quad \int_Q \mathfrak{m}_\alpha^+(E \cap X_\alpha) \, \mathfrak{q}(d\alpha) \\
&\geq \quad \int_Q \mathcal{I}_{\mathbb{S}^N}(\mathfrak{m}_\alpha(E \cap X_\alpha)) \, \mathfrak{q}(d\alpha) \ \text{by } 3.\text{+Smooth LGI in 1D} \\
&= \quad \int_Q \mathcal{I}_{\mathbb{S}^n}(\mathfrak{m}(E)) \, \mathfrak{q}(d\alpha) \ \text{by } 4. = \mathcal{I}_{\mathbb{S}^n}(\mathfrak{m}(E)).
\end{aligned}
$$

$\rightsquigarrow$ Lévy-Gromov inequality for e.n.b. $CD^*(N-1, N)$ spaces.

Quantitative Lévy-Gromov: one dimensional estimates

- Let (M^n, g) be with Ric$\geq n-1$ and let $\mathfrak{m} = vol_g/|M|$. Given $E \subset M$ with $\mathfrak{m}(E) = v \in (0,1)$, we have:

$$0 \leq \delta \quad := \mathfrak{m}^+(E) - \mathcal{I}_{\mathbb{S}^n}(v) \quad \text{"Lévy-Gromov isoperimetric deficit"}$$
$$\geq \int_Q \left(\mathfrak{m}^+_\alpha(E \cap X_\alpha) - \mathcal{I}_{\mathbb{S}^n}(v)\right) \mathfrak{q}(d\alpha) = \int_Q \delta_\alpha \mathfrak{q}(d\alpha).$$

- Since $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ is $CD(n-1, n)$ and $\mathfrak{m}_\alpha(E \cap X_\alpha) = \mathfrak{m}(E) = v$ (by 4.) $\Rightarrow 0 \leq \delta_\alpha := \mathfrak{m}^+_\alpha(E \cap X_\alpha) - \mathcal{I}_{\mathbb{S}^n}(v) =$ "1-dim Isop. Deficit"
- The 1-dim deficit δ_α controls $\pi - |X_\alpha|$:

$$\int_Q (\pi - |X_\alpha|)^n \, \mathfrak{q}(d\alpha) \leq C(n,v)\delta.$$

- RK: so far, also in the proof of Lévy Gromov, no role of OT: works for any 1-D localization.

Quantitative Lévy-Gromov: set of long rays

- Fix the set of long rays

$$Q_{long} := \{\alpha \in Q : (\pi - |X_\alpha|)^n \leq \sqrt{\delta}\} \simeq \{\alpha \in Q : \delta_\alpha \leq \sqrt{\delta}\},$$

  so that (from last slide) $\mathfrak{q}(Q_{long}) \geq 1 - C(n, v)\sqrt{\delta}$
- Problem: we know that most rays have length $\sim \pi$, but how do they combine together?
  Is there are a "common south/north pole"?
  NO for a general 1-D localization. However in our case
  Exploit the variational character of the localization via OT.

Quantitative Lévy-Gromov: structure of transport set

- Recall that for the set of long rays $\mathfrak{q}(Q_{long}) \geq 1 - C(n,v)\sqrt{\delta}$:

$$Q_{long} := \{\alpha \in Q : (\pi - |X_\alpha|)^n \leq \sqrt{\delta}\} \simeq \{\alpha \in Q : \delta_\alpha \leq \sqrt{\delta}\},$$

- From cyclical d-monotonicity of the transport set, we get

$$\begin{aligned} 2\pi \quad - \quad & \mathsf{d}(a(X_\alpha), b(X_\alpha)) - \mathsf{d}(a(X_{\bar\alpha}), b(X_{\bar\alpha})) \\ & \geq 2\pi - \mathsf{d}(a(X_\alpha), b(X_{\bar\alpha})) - \mathsf{d}(a(X_{\bar\alpha}), b(X_\alpha)) \end{aligned}$$

  Rearranging, for $\alpha, \bar\alpha \in Q_{long}$ gives

$$2\delta^{\frac{1}{2n}} \geq (\pi - \mathsf{d}(a(X_\alpha), b(X_{\bar\alpha}))) + (\pi - \mathsf{d}(a(X_{\bar\alpha}), b(X_\alpha)))$$

- Using Ric$\geq n-1$, setting $P_N := a(X_{\bar\alpha}), P_S := b(X_{\bar\alpha})$, we get

$$\mathsf{d}(a(X_\alpha), P_N) + \mathsf{d}(b(X_\alpha), P_S) \leq C(n,v)\delta^{\frac{1}{2n}}, \ \forall \alpha \in Q_{long}$$

Quantitative Lévy-Gromov: constructing the metric ball

- Using 1-dim (LGI), for $\alpha \in Q_{long}$, calling $E_\alpha := X_\alpha \cap E$ it holds
$$\min\{\mathfrak{m}_\alpha(E_\alpha \Delta [0, r_v]), \mathfrak{m}_\alpha(E_\alpha \Delta [|X_\alpha| - r_v, |X_\alpha|])\} \leq \delta_\alpha \leq \sqrt{\delta}$$
where r_v is s.t. $\mathfrak{m}_{\mathbb{S}^n}(B_{r_v}) = v$.
- So we can write $E = E_N \cup E_S \cup E_{err}$ with:
$\mathfrak{m}(E_{err}) \leq C(n, v)\sqrt{\delta}$,
$E_N := \{x \in E_\alpha \;\; E_\alpha \simeq [0, r_v]\}$,
$E_S := \{x \in E_\alpha \;\; E_\alpha \simeq [|X_\alpha| - r_v, |X_\alpha|]$
- Using relative isoperimetric inequality inside $B_\varepsilon(P_N)$ (or in $B_\varepsilon(P_S)$) with $\varepsilon \ll r_v$, we get
$$\min\{\mathfrak{m}(E_N), \mathfrak{m}(E_S)\} \leq C(n, v)\delta^{\frac{1}{n}}$$
- Putting all together:
$$\min\left\{\mathfrak{m}(E \Delta B_{r(v)}(P_N)), \mathfrak{m}(E \Delta B_{r(v)}(P_S))\right\} \leq C(n, v)\delta^{\frac{1}{n}}$$

Quantitative Obata's Theorem

- Given (M^n, g) with Ric$\geq n-1$, and $f : M \to \mathbb{R}$ with $\int_M f\,\mathfrak{m} = 0$, $\int_M f^2\,\mathfrak{m} = 1$, associate a 1D-localization:

$$\mathfrak{m} = \int_Q \mathfrak{m}_\alpha \mathfrak{q}(d\alpha), \ \int f\,\mathfrak{m}_\alpha = 0, \ (X, \mathsf{d}, \mathfrak{m}_\alpha) \in CD(n-1, n)$$

- Recalling that $\mathfrak{m}(M) = 1$, $\lambda_1(\mathbb{S}^n) = n$, let

$$\begin{aligned} 0 \leq \delta \quad &:= \quad \int_M (|\nabla f|^2 - n)\mathfrak{m} = \text{"Spectral deficit"} \\ &\geq \quad \int_Q \left(\int_{X_\alpha} \left((f|_{X_\alpha})' \right)^2 - n)\mathfrak{m}_\alpha \right) \mathfrak{q}(d\alpha) = \int_Q \delta_\alpha c_\alpha^2 \mathfrak{q}(d\alpha) \end{aligned}$$

  where $c_\alpha = \|f\|_{L^2(\mathfrak{m}_\alpha)}$.

- New difficulties:
  1) show that $c_\alpha \geq c > 0$ for "most" α, up to $\mathfrak{q}$-meas$\leq \delta$
  2) show that $c_\alpha \simeq c_{\bar{\alpha}}$ for "most" α, up to $\mathfrak{q}$-meas$\leq \delta$.

!!THANK YOU FOR THE ATTENTION!!

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