Example of last: $[0,1]\times\mathbb{R}^2/\sim_\theta$

Let $R(\theta)$ = rotation of $\mathbb{R}^2$ of angle theta,

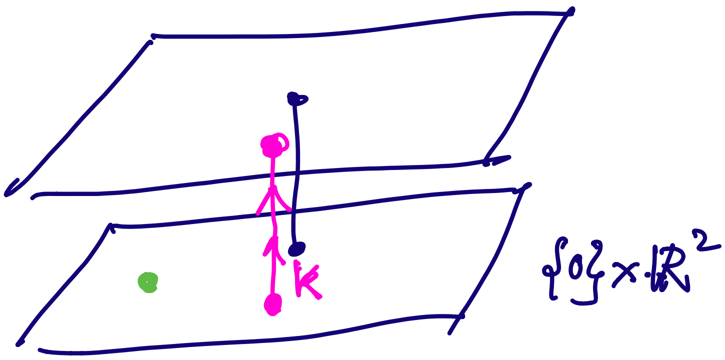
impose $\{0\}\times x \sim \{1\}\times R(\theta)x$.

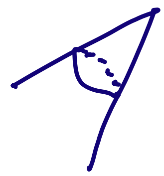
Let $w\in[0,1]$ be $[0,1]$-coord.

Let $K = \frac{\partial}{\partial w}$. Apply rescaling trick.

When $\theta = 0$, as $\delta\to 0$, GH limit = flat $\mathbb{R}^2$.

$\theta = 2\pi\, m_1/m_2$ as $\delta\to 0$, GH limit =

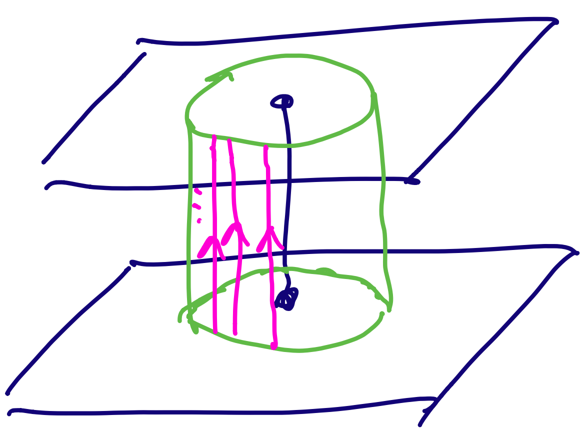




angle at tip $= \frac{2\pi}{m_2}$.

When $\theta\notin\mathbb{Q}$, as $\delta\to 0$,

GH limit = .



Today: F-structures of Cheeger-Gromov.

First define elementary F-structure on an open U as given by the following:

→ $\tilde{U} \to U$ finite normal covering of U with covering group Γ

→ $\exists\, k$, $\exists\, \rho : \Gamma \to \mathrm{Aut}(T^k)$ such that $\Gamma \ltimes_\rho T^k$ acts effectively on $\tilde{U}$.

Let's put elementary F-structures on flat mfds (will serve as a model for general case).

Recall:

→ Bieberbach: compact flat n-mfds are finitely covered by a flat torus and the covering degree is bounded by $\lambda(n)$.

→ Cheeger-Gromoll: if X^n is flat complete, then $X^n = X^\ell \times \mathbb{R}^k$ where X^ℓ has no $\mathbb{R}$-factor. Then X^ℓ contains a unique compact flat totally geodesic "soul" $S^m \hookrightarrow X^\ell$, such that X^ℓ is isometric to the total space of the normal bundle over S^m.

To find elementary F-structures:

→ on compact flat S^n, let $\tilde{S}^n \to S^n$ be the torus covering S^n, then a torus action can be given by the action of $\tilde{S}^n$ on itself. ↘ elem F-str

→ on non-compact flat mfd X^l (assume X^l has no $\mathbb{R}$-factor, X^l = normal bundle of S^m).
first consider $\tilde{X}^l \to X^l$ corresponding to $\tilde{S}^m \to S^m$.

- $\pi_1(X^l) = \pi_1(S^m)$ acts by isometries on $\mathbb{R}^l$
- $\overset{\vee}{\pi_1}(\tilde{X}^l) = \pi_1(\tilde{S}^m) =: \Delta \simeq \mathbb{Z}^m$ acts by isom on $\mathbb{R}^l$.

Any element $\varphi \in \Delta$ determines a 1-parameter family of isometries φ_t, for instance if in $\mathbb{R}^3$

$$\varphi = (w, R(\theta))$$

then define $\varphi_t = (tw, R(t\theta))$

⚠ $$\varphi = k\psi \in \Delta \not\Rightarrow \varphi_t = k\psi_t .$$

(for instance take $\varphi = 2\psi$, where $\varphi = (2, R(0))$
$\psi = (1, R(\pi))$

$$\Rightarrow \varphi_t = (2t, R(0)) \quad \psi_t = (t, R(\pi t)))$$

Now given $\varphi_1, \dots, \varphi_r \in \Delta \simeq \mathbb{Z}^m$, get $\varphi_{1,t}, \dots, \varphi_{r,t}$ families of isometries commuting, actually get a torus T^r-action $\tilde{X}^\ell = \overline{\mathbb{R}^\ell / \Delta}$.

(*) Suppose $\{\varphi_1, \dots, \varphi_r\}$ is invariant by conjugation by element of $\pi_1(X^\ell)$. Then there is an induced rep

$$\rho : \underbrace{\pi_1(X^\ell)/\Delta}_{\text{covering gp of } \tilde{X}^\ell \to X^\ell} \longrightarrow \mathrm{Aut}(T^r) = SL(n, \mathbb{Z}).$$

$\rightsquigarrow$ gives an elementary F-struct on X^ℓ.

Ex: $\underset{\rightsquigarrow}{}$ $\xi_\theta = [0,1] \times \mathbb{R}^2 / \sim_\theta$

- if $\theta = 0$, then an elem. F-struct exists, take S^1-action.
- if $\theta = \pi$, take $\varphi_1 = (1, R(\pi))$ isom of $\mathbb{R}^3$
  $\varphi_2 = 2\varphi_1$

then they are invariant by conjugation by $\pi_1(\xi_\pi) = \mathbb{Z}$.
(*)✓ $\Rightarrow$ elem F-structure on ξ_π and get a T^2-action induced by φ_1, φ_2.

One can translate that in terms of geodesic loops in X^ℓ: given geod. loops $\{\varphi_1, \dots, \varphi_r\} \hookrightarrow \Delta \simeq \mathbb{Z}^m$
if (*)✓ then get elem. F-structure

—

C-G: $(*)$ can be satisfied by taking
"good loops in X^e with $\text{small}_{\dim}$ length and $\text{small}_{\dim}$ holonomy".

Why is this procedure useful:
Given $p \in (M,g)$ $|\sec_g| \leq 1$, injrad $\ll 1$,
blow-up the metric at p so that injrad ≥ 1 in a large neighborhood of p, then this large neighb is close to a flat complete mfd so it carries automatically an elementary F-structure.

We want to glue together these elementary F-structures.
Define F-structures as follows on M:
M is covered by a finite cover of open sets U_α, each U_α admits an elementary F-structure (i.e. $\exists\, \tilde{U}_\alpha \to U_\alpha$, $\exists$ effective action of $\Gamma \ltimes T^{k_\alpha}$ on $\tilde{U}_\alpha$), and the following compatibility condition is satisfied:

if $U_\alpha \cap U_\beta \neq \emptyset$,
$\exists$ $\tilde{U}_{\alpha\beta}$ common covering of $\tilde{U}_\alpha$, $\tilde{U}_\beta$ such that

actions of $T^{k_\alpha}, T^{k_\beta}$ lift to actions on $\tilde{U}_{\alpha\beta}$ by $T^{k_\alpha}, T^{k_\beta}$, and either $T^{k_\alpha} \hookrightarrow T^{k_\beta}$ or vice versa.

respects the action.

Ex: look $T^2 \times (0,1)$

$U_1 = T^2 \times (0, \frac{2}{3})$, $\quad U_2 = T^2 \times (\frac{1}{3}, 1)$

$\hookrightarrow \exists\ T^2$-action $\qquad \hookrightarrow \exists\ S^1$-action

That gives an F-structure.

The F-structure is said to be of positive rank if orbits of the F-structure have > 0 dimension.

Thm: (CG) $\exists\ \delta_n > 0$, $\forall\ (M,g)$ $\quad |\sec_g| \leq 1$, $\quad \text{injrad} \leq \delta_n$

then M carries an F-structure of positive rank.

(Rmk: definition of F-struct doesn't need Riemannian geom)

Pf (sketch): Previous algo + blow-up argument give near any point an elem F-struct.

To "glue" these together, use geod loops of "small length, small holonomy":

$x \mapsto \{\gamma_1^x, \dots, \gamma_{N_x}^x\}$ those geod loops

$y \mapsto \{\gamma_1^y, \dots, \gamma_{N_y}^y\}$

Now if x close to y, then $\exists$ natural inclusion of 1st family into 2nd family or vice versa.

$\boxtimes$

Thm: (CG) "Conversely" if M has an F-structure of positive rank, $\exists \{g_t\}_{t\in[0,1]}$ continuous family of metrics collapsing ($|\sec_t| \leq C$, injrad $\to 0$).

In fact if (M,g) has $|\sec| \leq 1$, injrad $\leq \varepsilon_n$, then $\exists \{g_t\}$, with $g_1 = g$.

⚠ Here the sectional upper bound C depends on g !

Pf: Idea = apply the "Killing field rescaling trick".

Easy case: consider elem-F-struct. such that the local torus action is "free", i.e. the torus orbits all have same dimension as the torus, (we say that the elem. F-structure is "polarized")

$\rightsquigarrow$ Apply "rescaling trick" by rescaling these torus orbits by δ^2

Less easy case: the F-structure is made of polarized elem. F-structures glued together,

$\rightsquigarrow$ Apply "rescaling trick" to each elem. F-structure and then use cut-off function arguments.

Ex: $T^2 \times [0,1]$ with flat metric g

let $U_1 = T^2 \times [0, \frac{2}{3})$, $U_2 = T^2 \times (\frac{1}{3}, 1]$

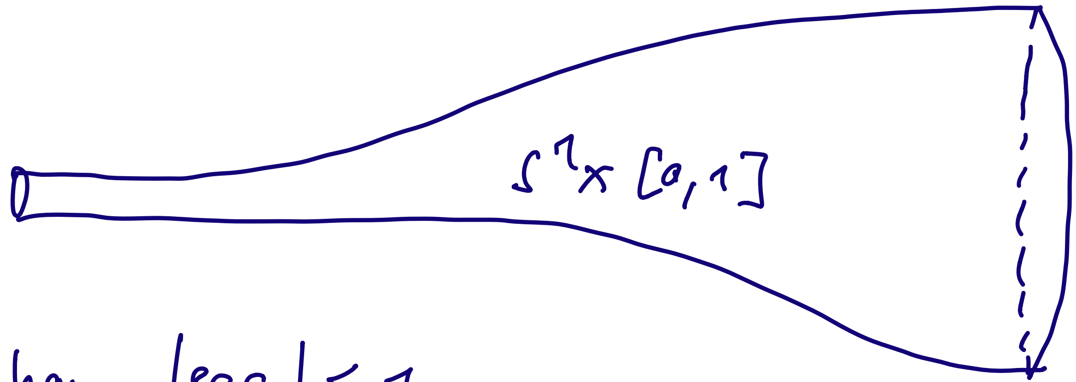
$\hookrightarrow T^2$-action

$\hookrightarrow S^1$-action (say the first S^1 in T^2)

How do we collapse g with resp. to that F-struct?

Put a metric on $T^2 \times [0,1]$ called g_δ,

$$(T^2 \times [0,1], g_\delta) = S^1(\delta) \times (S^1 \times [0,1], h_\delta)$$

h_δ : 

where h_δ has $|\sec| \leq 1$

$$(S^1 \times \{0\}, h_\delta) \underset{\text{isom}}{=} S^1(\delta)$$
$$(S^1 \times \{1\}, h_\delta) \underset{\text{isom}}{=} S^1(1).$$

Note $\operatorname{diam}(T^2 \times [0,1], g_\delta) \xrightarrow[\delta \to 0]{} \infty$.

<u>Rmk:</u> when collapsing metrics with F-structure, in general need to have $\operatorname{diam} g_\delta \longrightarrow \infty$.