What is the (essential) minimal volume?

$|\sec| \leq 1 \longleftrightarrow$ topology

$\longleftrightarrow$ Ricci

$\longleftrightarrow$ open questions

Goals: → introduce MinVol, ess-MinVol,

→ collapsing theory of Cheeger-Fukaya-Gromov

<u>Plan</u>: I. MinVol, ∞-MinVol, results

II. Collapsing theory: F-structures, collapsing construction

III. Collapsing theory: N-structures

IV. Collapsing construction bis, applications.

M closed manifold of dim n.

$\mathcal{M}_{|\sec| \leq 1}(M) = \{ (M,g) \, ; \, |\sec_g| \leq 1 \}$.

Definition: • $\mathrm{MinVol}(M) := \inf \{ \mathrm{Vol}(M,g) \, ; \, (M,g) \in \mathcal{M}_{|\sec| \leq 1}(M) \}$
(Gromov)

• $\forall \delta > 0$, denote by $M^{(g)}_{>\delta} := \{ x \in M ; \ \mathrm{injrad}_x > \delta \text{ wrt } g \}$.

$$\text{ess-MinVol}(M) := \lim_{\delta \to 0} \inf \{ \mathrm{Vol}(M^{(g)}_{>\delta}, g) \, ; \, (M,g) \in \mathcal{M}_{|\sec| \leq 1}(M) \}$$

By def ess MinVol $\leq$ MinVol.

Let's rewrite ess. MinVol as an infimum of volume.

Recall: Cheeger: if $|\sec| \leq 1$, injrad $> \delta$

then (pointed) $C^{1,\alpha}$ compactness.

Next define a weak closure of $\mathcal{M}_{|\sec| \leq 1}(M)$:

let $(M, g_i) \in \mathcal{M}_{|\sec| \leq 1}(M)$. It converges weakly if:

$\forall \delta > 0$, $\exists$ Q_δ, $\exists$ $x_{i,1}, \dots, x_{i,Q_\delta}$,

a) $\lim_{i\to\infty} \mathrm{dist}_{g_i}(x_{i,k}, x_{i,l}) = \infty \qquad \forall\, k \neq l \in \{1, \dots Q_\delta\}$

b) $(M^{(g_i)}_{>\delta}, \mathrm{dist}_{g_i}, x_{k,i}) \xrightarrow[\rho GH]{} (Y_k^\delta, h_k^\delta, y_k^\delta)$

$\forall\, k \in \{1, \dots, Q_\delta\}$

c) $\forall$ sequence of points $p_i \in M^{(g_i)}_\delta$,

$\mathrm{dist}_{g_i}(p_i, \{x_{k,i}\}_{k=1}^{Q_\delta})$ bounded as $i \to \infty$.

In that case, the "weak limit" of (M, g_i) is

$$\bigcup_{\delta>0} \bigsqcup_{k=1}^{Q_\delta} (Y_k^\delta, h_k^\delta, y_k^\delta).$$

(can have ∞ many connected components)

Use this notion of weak limit, define $\overline{\mathcal{M}}^{\text{weak}}_{|\sec| \leq 1}(M)$ as the weak limits of $g_i \in \mathcal{M}_{|\sec| \leq 1}(M)$.

Now, claim: $\text{ess-MinVol}(M) = \inf \left\{ \text{Vol}(N,h) \,; \, (N,h) \in \overline{\mathcal{M}}^{\text{weak}}_{|\sec| \leq 1}(M) \right\}$.

Moreover (because of Cheeger):

$\text{ess MinVol}(M) = \text{Vol}(M_\infty, g_\infty)$ where $(M_\infty, g_\infty) \in \overline{\mathcal{M}}^{\text{weak}}_{|\sec| \leq 1}(M)$.

$\hookrightarrow$ complete $C^{1,\alpha}$ metric, M_∞ smooth mfd.

Some important results:

There are very few general topol lower bound for MinVol (or ess-MinVol).

- "Trivial bound":

$\forall n$, $\forall$ characteristic number $\mathcal{P}$,

$$\mathrm{MinVol}(M) \geq C_{n,\mathcal{P}} \quad \mathcal{P}(M)$$

$$\mathcal{P}(M) = \int \mathcal{P}(\text{curvature}) \leq C_{n,\mathcal{P}} \int 1$$

$$\text{ess-MinVol}(M) \geq C_n \quad \chi(M)$$

$\chi(M) \longrightarrow$ euler characteristic

- For the 2^{nd} lower bound, need the "simplicial volume" $\|M\|$.

$$\|M\| := \inf \left\{ \sum_{i=1}^{\ell} |r_i| \; ; \; \exists \text{ singular simplices } \sigma_1, \dots, \sigma_\ell : \underbrace{\Delta^n}_{\text{standard simplex}} \longrightarrow M \text{ and } \sum_{i=1}^{\ell} r_i \sigma_i \text{ is a cycle representing } [M] \in H_n(M;\mathbb{Z}) \right\}$$

(Gromov)

(M oriented)

⚠ $r_i \in \mathbb{R}$

$\|M\|$ in some sense is the "minimal amount" of simplices needed to cover M. "Cycle" means $\partial \sum r_i \sigma_i = \sum r_i \partial \sigma_i = 0$.

"Easy property" : if $M_1 \xrightarrow{f} M_2$ has degree d,

then $\|M_1\| \geq d\,\|M_2\|$.

(indeed, if $\sum r_i \sigma_i$ represents $[M_1]$, it is sent via f to $\sum r_i f_*(\sigma_i)$ which represents $d[M_2]$.

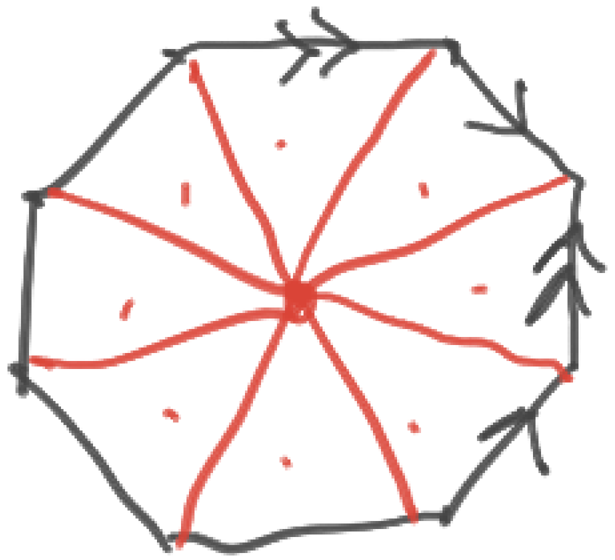
$\Rightarrow \frac{1}{d} \sum r_i f_*(\sigma_i)$ represent $[M_2]$,

but $\|M_2\| \leq \frac{1}{d} \underbrace{\sum |r_i|}_{\text{take it close to } \|M_1\|}$.)

• As a consequence: if Σ_γ is the surface of genus γ, then
$$\|\Sigma_\gamma\| = \begin{cases} 0 & \text{if } \gamma = 0, 1 \\ 4\cancel{g}\,\gamma - 4 = 2|\chi| & \text{if } \gamma > 1 \end{cases}$$

Pf: • for sphere / torus which have self-maps of degree $\neq 0$, apply "easy property", get $\|\ \| = 0$.

• For $\gamma > 1$: First check that $\|\Sigma_\gamma\| \leq \underline{2|\chi| + 4}$



$\gamma = 2$

Then look at a degree d-cover $\Sigma_{\tilde{\gamma}} \to \Sigma_\gamma$, then by "easy property":
$$\|\Sigma_{\tilde{\gamma}}\| \geq d \, \|\Sigma_\gamma\|$$

But $\|\Sigma_g\| \leqslant 2|\chi(\Sigma_{\tilde g})| + 4 = 2d|\chi(\Sigma_g)| + 4$

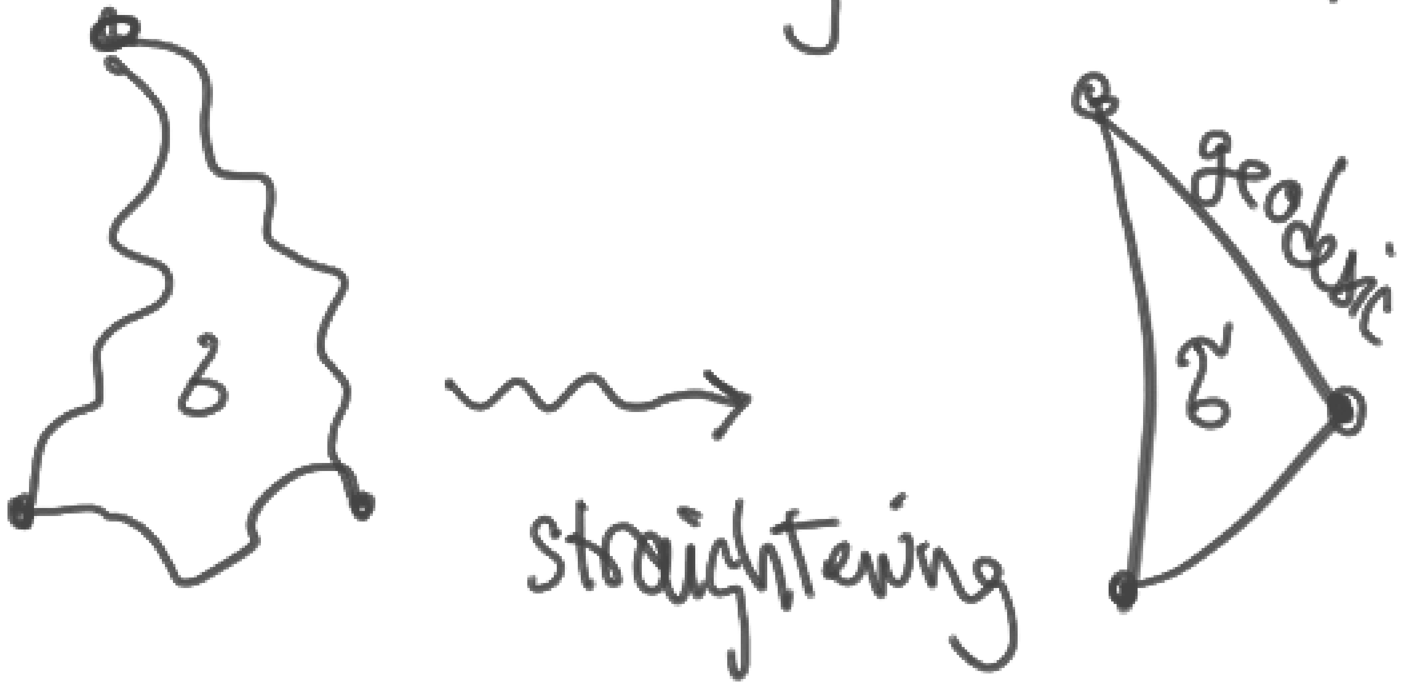
Now send $d \to \infty$: get $\|\Sigma_g\| \leqslant 2|\chi(\Sigma_g)|$. ✓

Next: $\|\Sigma_g\| \geqslant 2|\chi(\Sigma_g)|$, indeed

given $\sum r_i \sigma_i$ cycle rep. $[\Sigma_g]$,

put hyperbolic metric on Σ_g , you can

"straighten" σ_i to $\tilde{\sigma}_i$, then



$$2\pi|\chi(\Sigma_g)| = \sum r_i \,\mathrm{Vol}(\sigma_i)$$
$$\leqslant \sum |r_i| \,\mathrm{Vol}(\sigma_i)$$
$$\underset{\text{hyperb.}}{\leqslant} \sum |r_i| \,\pi .$$

Gromov : $\exists\, \delta_n > 0$, $\exists\, C_n$, $\forall g$ metric in $\mathcal{M}_{|\mathrm{sec}| \leq 1}(M)$

then $\mathrm{Vol}(M^{(g)}_{\geq \delta_n}) \geqslant C_n \|M\|$.

[in particular : $\mathrm{MinVol}(M) \geq \mathrm{ess\,MinVol}(M) \geqslant C_n \|M\|$.]

- Besson - Courtois - Gallot : if X_{hyp} is a closed hyperbolic manifold then $\mathrm{MinVol}(X_{hyp}) =$ hyperbolic volume of X_{hyp}.

Rmk : we'll see that it also holds for the ess. MinVol.

Rmk: in dim 2, 3 ess MinVol = MinVol.

Rmk: by Bessières, Kotschik, MinVol (and also ess-MinVol) depends on the smooth structure of M.

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2 conjectures:

Gap conjecture (Gromov): $\exists\, \varepsilon_n > 0$, $\forall M$ with $\mathrm{MinVol}(M) \leq \varepsilon_n$ then $\mathrm{MinVol}(M) = 0$.

(know in dim ≤ 4)
Rong.

We will see that by Cheeger-Gromov, the Gap conjecture is true for ex-MinVol.

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Conjecture: in dim 4, $\forall$ N^4 closed, $\forall$ genus $\gamma > 1$

then $(\text{ex})\text{-MinVol}(N \# S^2 \times \Sigma_\gamma) \geq c_n \gamma$.

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$\rightsquigarrow$ Studying MinVol / ex-MinVol

$\leftrightarrow$ studying "closures" of $\mathcal{M}_{|\sec| \leq 1}$

$\leftrightarrow$ need to understand thin part $\{\text{injrad} \leq \delta_n\}$.

II. Collapsing theory: F-structures, collapsing construction.

"g_i collapses": $|\sec_{g_i}| \leq C$, $\mathrm{injrad}_{g_i} \longrightarrow 0$ everywhere.

- Basic example of collapsing:

"Rescaling along Killing fields trick".

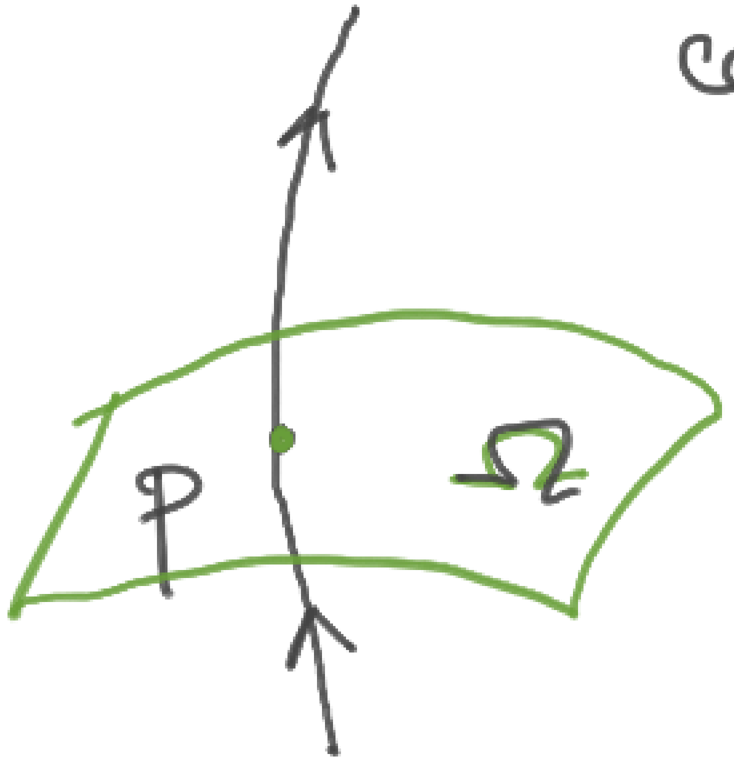
Let (M,g) cpct, K = Killing field on M, generates an $\mathbb{R}$-action.

Write $g = \underbrace{g_T}_{\text{tangent to } K} + \underbrace{g_P}_{\text{orthogonal to } K}$. Define $g_\delta = \delta^2 g_T + g_P$.

$1 \geq \delta > 0$

Claim: as $\delta \to 0$, $|\sec_{g_\delta}| \leq C(g)$ independent of δ.
and $\text{injrad} \xrightarrow[\delta \to 0]{} 0$ (collapsing).

Pf: $p \in M$, let Ω be a transversal at p (Ω = open codim 1 submfd embedded, going through p, transversal to the $\mathbb{R}$-fibers.)



Let $(y^1, \dots, y^r)$ coord on Ω, $p = (0, \dots, 0)$ and let x be the $\mathbb{R}$-coord corresponding to K. $x = 0$ on Ω. Extend $(y^1, \dots, y^r)$ to whole neighb of p, by projecting on Ω along the $\mathbb{R}$-fibers.

Here $\frac{\partial}{\partial x} \parallel \mathbb{R}$-fiber,

write $\frac{\partial}{\partial y^i} = \underset{\substack{\text{tangent} \\ \text{to } K}}{X_i} + \underset{\substack{\text{orthogonal} \\ \text{to } K}}{V_i}$

then $g = \begin{pmatrix} A & B \\ B & C+D \end{pmatrix}$ where

$A = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \rangle_g$, $B_{1\ell} = \langle \frac{\partial}{\partial x}, X_\ell \rangle_g$, $C_{m\ell} = \langle X_m, X_\ell \rangle_g$
$D_{m\ell} = \langle V_m, V_\ell \rangle_g$

$g_\delta = \begin{pmatrix} \delta^2 A & \delta^2 B \\ \delta^2 B & \delta^2 C + D \end{pmatrix}$, change coord $u = \delta x$, $\partial/\partial u = \frac{1}{\delta}\, \partial/\partial x$ get $g_\delta = \begin{pmatrix} A & \delta B \\ \delta B & \delta^2 C + D \end{pmatrix}$.

$g_\delta \xrightarrow[\delta \to 0]{C^\infty} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$

and remember A, B, C, D only depend on the y coordinates because $K =$ Killing

$\Rightarrow$ g_δ has $|\sec_{g_\delta}| \leq C(g)$, and

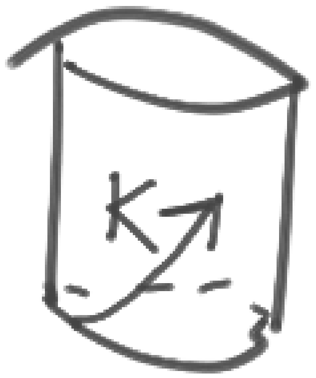
injrad $\longrightarrow 0$ because of cpctness of the $\mathbb{R}$-orbits (M cpct).

Rmk: this trick extends to case where we have $K_1, \dots, K_e$ Killing fields that commuting.

Example: look at flat $([0,1]\times\mathbb{R}^2, h)$,
let $B(\theta)$ be the rotation of $\mathbb{R}^2$ with angle θ,
Consider $[0,1]\times\mathbb{R}^2 / \sim_\theta$ $:=$ glue $\{0\}\times\mathbb{R}^2$ to $\{1\}\times\mathbb{R}^2$ with $B(\theta)$.

($\underset{\text{diffeo}}{\cong}$ to $S^1\times\mathbb{R}^2$)

Apply rescaling trick to $\frac{\partial}{\partial t}\left[B(t\theta)\cdot + t\right]$.



If $\theta = 0$, then rescale, as f
GH limit $= \mathbb{R}^2$ flat