Summary of last time:

→ definition of F-structures

CG1990 → Existence: collapsed mfds have F-structures of >0 rank

CG1986 → Collapsing construction: Conversely if $\exists$ F-struct. of >0 rank then $\exists$ $\{g_t\}_{t\in(0,1]}$ collapsing as $t\to 0$.

Comments:

- think of F-structure as torus actions on U_α, $\{U_\alpha\}$ open cover of M
- think of F-orbits as the "orbits of the local torus action
- think of ">0 rank" as "effective action".
- think of "polarized" as "free action".

- Given F-structure on M. M is a union of F-orbits, each orbits $\approx {}_{\Gamma}\backslash T^k$ finite quotient of a torus
- If M admits an F-structure of >0 rank : $\chi(M) = 0$.
  Euler

Pf of collapsing construction: Start with a metric g invariant for F-structure. (by an averaging argument, g exists)

easy cases: polarized F-structures (= union of elem polarized F-struct) then apply rescaling trick to each elem F-struct, + use cut-off functions.

concretely, take U_α, let $f_\alpha : U_\alpha \longrightarrow [\frac{1}{2}, 1]$ F-invariant

$f_\alpha \equiv 1$ near ∂U_α, $f_\alpha \equiv \frac{1}{2}$ inside $U_\alpha \setminus$ (neigh of ∂U_α),

set $\rho_\alpha := \delta^{\log f_\alpha / \log 1/2}$.

set $g_\delta := \log^2 \delta \left[\rho_\alpha^2 g_T + g_P \right]$.

check $|\sec_{g_\delta}| \leq C(g)$ $\quad \mathrm{injrad}_{g_\delta} \longrightarrow 0$.

Difficult case: elem F-struct non-polarized

Ex: non-polarized F-structure exist

$S^1 \times \mathbb{R}^2$, T^2-action given by

$$\underset{\substack{\cap \\ T^2}}{(s,t)} \cdot \underbrace{\left(e^{2\pi i\alpha}, x\right)}_{\in S^1 \times \mathbb{R}^2} = \left(e^{2\pi i(\alpha+s)}, R(2\pi t)x\right).$$

Not free (so not polarized).

Here clearly $\exists$ S^1-action free.

Ex: In general one <u>cannot</u> make a non-polarized F-str. polarized.

$$\mathcal{E}_\theta = {[0,1] \times \mathbb{R}^2}/{\sim_\theta} \quad \text{where } \{0\} \times x \sim_\theta \{1\} \times \underbrace{R(\theta)}_{\text{rotation}} x$$

Consider $\{(\theta, \mathcal{E}_\theta)\ ;\ \theta \in [0, 2\pi]\}$

$\mathcal{E}_0$ is isometric to $\mathcal{E}_1$: $\exists$ $f : \mathcal{E}_0 \longrightarrow \mathcal{E}_1$ isom.

Look at closed 4-mfd M^4:

$$M = \{ (\theta, \mathcal{E}_\theta) ; \theta \in [0, 2\pi] \} \Big/ \sim_f \qquad (0, x) \sim_f (1, f(x)).$$

this manifold is <u>not</u> diffeo to $S^1 \times (S^1 \times \mathbb{R}^2)$.

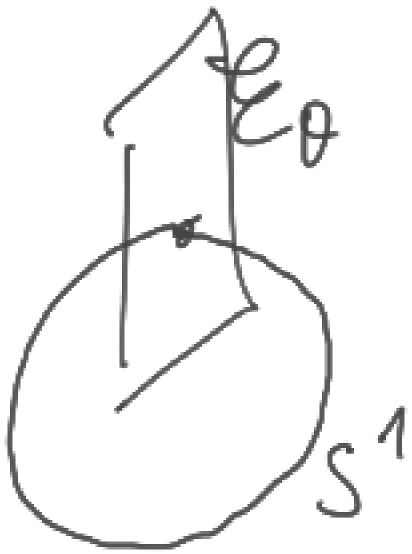
In fact if one looks at $C_\theta := [0,1] \times \partial B(1) \Big/ \sim_\theta \subset \mathcal{E}_\theta$

then $\{ (\theta, C_\theta), \theta \in [0, 2\pi] \} \Big/ \sim_f \subset M$ is a 3-nilmanifold which is a T^2-bundle over S^1 with holonomy given by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

M^4 carries an $\mathcal{F}$-structure of >0 rank:

$M^4 = U_1 \cup U_2$, each U_i carries a T^2 action, take U_i of the form $U_i = \{(\theta, \xi_\theta), \theta \in$ 1/2 half of $S^1\}$.

Claim: one cannot make that F-structure polarized, because of the non-trivial holonomy.



To collapse, the idea is to choose smoothly an $\mathbb{R}$-action free in each ξ_θ. The vector field induced by this $\mathbb{R}$-action is called K. $K \neq 0$, K is a Killing field in each ξ_θ, but K is not a Killing.

Now try to apply rescaling trick to g using K.

idea: contract metric in direction of K, and <u>expand</u> metric in direction orthogonal to $\{\theta\} \times \Xi_\theta$, to compensate for the fact that K is not a Killing.

i.e. if $g = \underbrace{g_\perp}_{\substack{\text{orthogonal} \\ \text{to } \{\theta\} \times \Xi_\theta}} + \underbrace{(g_T + g_P)}_{\text{part tangent to } \{\theta\} \times \Xi_\theta}$

$$g_\delta := \frac{1}{\delta^2} g_\perp + \delta^2 g_T + g_P .$$

$\rightsquigarrow$ check that g_δ collapses (see p. 330 CG 1986).

In general, to apply that idea to a non-polarized F-structure, need to stratify M into strata$_i$ = {F-orbits of dim = i}. (delicate).

In conclusion:

$$|\sec_{g_\delta}| \leq C(g)$$

$$\mathrm{injrad}_{g_\delta} \longrightarrow 0$$

$$\text{but } \mathrm{volume}(g_\delta) \not\longrightarrow 0.$$

$\square$

Important comment: Given g with $|\sec| \leq 1$, injrad $\leq \delta_n$, $\exists$ F-struct by CG associated to g, which is living on the <u>scale</u> of <u>injrad</u>$_g$. (F-orbits found have size $\approx$ injrad).

Applications of existence / collaps. constr.

1^{st} app: Recall $\mathrm{MinVol}(M) := \inf\{ \mathrm{Vol}(M,g) \,,\, |\sec_g| \leq 1 \}$

$\text{ess-MinVol}(M) := \lim_{\delta \to 0} \inf \{ \mathrm{Vol}(M^{(g)}_{>\delta} ; g) ; |\sec_g| \leq 1 \}$

Remember Gap conj: $\mathrm{MinVol} \leq \varepsilon_n \Rightarrow \mathrm{MinVol} = 0$.

Corollary of CG: $\exists\, \varepsilon_n > 0$, $\text{ess-MinVol} \leq \varepsilon_n \Rightarrow \text{ess-MinVol} = 0$

<u>Pf</u>: supp $\text{ess-MinVol}(M) \leq \cancel{\delta_n/2}\ \varepsilon_n$ ~~(δ_n as in thm of CG)~~

$\forall \delta\ \exists g\ |\sec_g| \leq 1$, $\mathrm{Vol}(M^{(g)}_{>\delta}, g) \leq \varepsilon_n$

so injrad_g cannot be much larger than δ, so it is small everywhere (say $< \delta_n$)

By CG, $\exists$ F-struct.

By CG collapsing const, $\exists (g_t)$ on M collapsing, $\Rightarrow \text{ess-MinVol}(M) = 0$. ☐

2^{nd} appl: Jankiewicz's example (see CG 1986);

$\exists\ M^4$ with $\mathrm{MinVol}(M) > 0$, ess-$\mathrm{MinVol}(M) = 0$.

Pf: $T^3 = (e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3})$ acts on $\mathbb{C}^3$

D = diagonal in $T^3 = (e^{i\theta}, e^{i\theta}, e^{i\theta}) \subset T^3$

T^3/D acts on $\mathbb{C}P(2) = (z_1, z_2, z_3)/D$ where $\sum |z_i|^2 = 1$.

This action has 3 fixed points $(1,0,0)$ $(0,1,0)$ $(0,0,1)$.

Let $\Sigma_1 = \mathbb{C}P(2) \setminus \{$balls around those 3 fixed points$\}$,

let Σ_2 = copie Σ_1. Glue Σ_1 to Σ_2 along bdies

using for instance around $(1,0,0)$: $(1, w_2, w_3) \sim (1, \overline{w_2}, w_3)$

get M^4: M^4 is oriented, has T^2-action on $\Sigma_i \Rightarrow$ has an

F-struct of >0 rank. $\underset{CG}{\Longrightarrow}$ ess-$\mathrm{MinVol} = 0$.

But $\underset{\text{signature}}{\sigma}(M) = 2\sigma(\Sigma_1) = 2[\sigma(\mathbb{C}P(2)) - 3\sigma(\text{4-ball})] = 2\sigma(\mathbb{C}P(2)) = 2 > 0$

$\Rightarrow \mathrm{MinVol}(M) > 0$. ☐

III. Collapsing theory : N-structures.

Recall : F-structures only see things at level of injrad.

Pb: What's happening at scale 1 ??

Trivial exampl: $T^N = S^1 \times \dots \times S^1$, rescale each S^1 at different speed. Look at $S^1(\varepsilon) \times \dots \times S^1(\varepsilon^N)$ as $\varepsilon \to 0$.

Nilmanifold exampl: Lie group $N := \left\{ \begin{pmatrix} 1 & & u_{ij} \\ 0 & \ddots & \\ 0 & 0 & 1 \end{pmatrix} \right\}$

rmk : N is nilpotent is the sense that $[[[N,N],N],\dots] = \{1\}$

Lie alg $\mathfrak{g} = \left\{ \begin{pmatrix} 0 & & a_{ij} \\ & \ddots & \\ 0 & & 0 \end{pmatrix} ; a_{ij} \in \mathbb{R} \quad 1 \leq i < j \leq n \right\}$.

For $A \in \mathfrak{g}$, set $\|A\|_\delta^2 := \sum_{i<j} a_{ij}^2 \; \delta^{2(j-i)}$ $(\delta > 0)$

$\rightsquigarrow$ Get left invariant metric on N. For such a metric, sectional curv. is determine by $[\cdot,\cdot]$ Lie Bracket on $\mathfrak{g}$. (see Milnor)

But here $\|[A,B]\|_\delta^2 \leq C_n \|A\|_\delta^2 \|B\|_\delta^2$

Indeed for $AB = (a_{ij})(b_{kl}) = a_{im} b_{mj}$

$$\|AB\|_\delta^2 = \sum_{j>i} (a_{im} b_{mj})^2 \; \delta^{2(j-i)}$$

$$= \sum_{j>i} (a_{im})^2 \delta^{2(m-i)} \; (b_{mj})^2 \delta^{2(j-m)}$$

$\Rightarrow |\sec| \leq 1.$ $\qquad \leq c_n \|A\|_\delta^2 \|B\|_\delta^2$

$\Rightarrow |\sec| \leq C_n'$. As $\delta \to 0$, get collapsing metric on for instance $\Gamma \backslash N$ where Γ = cocompact lattice of N given by integer subgroup of N.

Then $\operatorname{diam}(\Gamma \backslash N) \xrightarrow[\delta \to 0]{} 0$

<u>Rmk</u>: $|\sec| \leq C_n'$, $\operatorname{diam} \to 0$ ($\Gamma \backslash N$ is called "almost flat")

$\Gamma \backslash N$ but cannot be flat due to Bieberbach.

Gromov, : Any almost flat mfd (i.e. g_i $|\sec_{g_i}| \leq C$, $\operatorname{diam} g_i \to 0$)
(Ruh) is finitely covered by a nilmanifold $\Gamma \backslash N$.

F-structures

tori
abelian
Bieberbach

N-structures

$\Gamma \backslash N$ nilmanifold
nilpotent
Gromov