III. (continued)

Recall : • F-structures $\approx$ local torus actions. < existence / collapsing const.
• example of nilpotent collapsing
• Gromov's almost flat mfd thm

M almost flat if $\exists\, g_i$ on M, $|\sec_{g_i}| \leq 1$, $\operatorname{diam}_{g_i} \to 0$.
M' nilmanifold if $M' \underset{\text{diffeo}}{\simeq} \Gamma\backslash N$, where N is a nilpotent Lie group (i.e. $[[[N,N]N]\cdots N] = \{1\}$), $\Gamma < N$ cocompact.

Gromov (Ruh) : Any almost flat manifold is diffeo to a finite quotient of a nilmanifold $\Gamma\backslash N$.

Rmk: ref: Buser & Karcher,
$\mathrm{Ric} \geq -(n-1) \rightsquigarrow$ generalization on the level of π_1
(Kapovich - Wilking, Courtois)

—

Gromov's thm can be generalized to parametrized, see Fukaya 87 88 89:
If (M^n, g), (Y^m, h) are Riemannian mfds, $|\sec_g| \leq 1$,
$d_{GH}((M,g), (Y,h)) \leq \varepsilon_n$ then

- $\exists$ fibration map f $\quad Z^{n-m} \to M^n \xrightarrow{f} Y$
- The fiber Z is an almost flat mfd. (= finite quotient of nilmanifold)

Rmk: if sequence (M^n, g_i), $|\sec_{g_i}| \leq 1$, take pointed GH limit of (M, g_i), get (X, d).

~~But~~ X doesn't need to ~~be~~ an orbifold... However,

near any $x \in X$, $\exists\, U \ni x$, $U \underset{\text{homeo}}{\simeq} V/G$,

where V = neigh of 0 in $\mathbb{R}^n$, G = Lie gp of transformations.

Intuition: Take $U_i \subset (M, g_i) \to U \subset X$,

look at $\tilde{U}_i \to U_i$ universal cover, induced metric $\tilde{g}_i$,

and the injectivity radius of $\tilde{U}_i$ is $> c > 0$.

Consider $\pi_1(U_i) =: G_i$, it acts ~~by~~ isometries on $\tilde{U}_i$.

Take a "limit" of those G_i, get G.

(consider elements of G_i as Lipschitz maps, use Arzela-Ascoli, ...)

G is in fact a Lie group and $V \text{"="} \text{"}\lim \tilde{U}_i\text{"}$,

$U \simeq V/G$

<u>Ex</u>: $\mathbb{Z} \curvearrowright \mathbb{R}^2$, $\underset{\mathbb{Z}}{k} \cdot \underset{\in \mathbb{R}^2}{x} := x + k\eta e$

$e \in \mathbb{R}^2$

Here as $\eta \to 0$, $\mathbb{Z} \text{"}\hookrightarrow\text{"} \mathbb{R}$.

$\eta \in (0,1]$

How to apply fibration thm of Fukaya??

Idea of Fukaya: look at the orthonormal frame bundle!

Recall: if (M,g) $|\sec_g| \leq 1$,

FM frame bundle

FM = bundle of orthonormal basis $(e_1, \dots, e_n)_p$
= an $O(n)$ - principle bundle.

g lifts to $\hat{g}$ on FM using Levi-Civita connection,
$(FM, \hat{g}) \xrightarrow{\pi} (M, g)$ Riem submersion, ($O(n)$-fibers sent to points $\in M$),
$O(n)$ acts by isometries on FM, $|\sec_{\hat{g}}| \leq C_n$.

Revised Rmk: Given (M, g_i) $|\sec_{g_i}| \leq 1$, suppose g_i are A-regular
(i.e. $\forall j$ $\|\nabla^{(j)} g_i\| \leq A^{(j)}$).
Get $(\hat{X}, \hat{d}) :=$ pGH limit of $(FM, \hat{g}_i)$.
Then $\hat{X}$ = smooth mfd
$\hat{d}$ = smooth metric

$$\begin{array}{ccc} (FM, \hat{g}_i) & \xrightarrow{\quad \pi_i \quad} & (M, g_i) \\ \text{pGH converges} \downarrow & & \downarrow \text{pGH converges} \\ (\hat{X}, \hat{d}) & \xrightarrow{\quad \pi_\infty \quad} & (X, d) \end{array}$$

$O(n) \curvearrowright (\hat{X}, \hat{d})$ by isom, π_∞ is given by $\hat{X}/O(n)$.

By previous remark: X is locally $\approx V/G$

$\Downarrow$

$\hat{X}$ is locally $\approx \hat{V}/G$

But G comes from isometries of $(\tilde{U}_i, g_i)$, and non-trivial isometries of $(\tilde{U}_i, g_i)$ lift to isom of $(F\tilde{U}_i, g_i)$ without fixed point.

So G acts freely on $\hat{V}$ "=" limit of $F\tilde{U}_i$.

$\Rightarrow$ $\hat{V}/G$ is a smooth mfd by slice thm.

$\Rightarrow$ can apply the fibration thm of Fukaya.

CFG 92 : using these ideas, get equivalent statement:

$\exists\, \delta_n, \rho_n > 0$, Given (M,g) $|\sec_g| \leq 1$, g is A-regular

if $\mathrm{injrad}_g < \delta_n$, $\forall x \in M$, $\exists\, (Y,h)$, $\exists\, U$ neighb

of x containing $B(x, \rho_n)$ such that:

- $\exists$ fibration map $\quad Z \to FU \xrightarrow{f} (Y, h)$
- $Z \underset{\text{diffeo}}{\simeq} T^N$
- the fibration map is $O(n)$ - invariant
- f is almost a Riemannian submersion, f is B-regular
  $(B = B(n, A))$

$\leadsto$ We can take that property as our definition for "elementary N-structure" on U ($\supset B(x, \rho_n)$).

Then an N-structure is defined as a finite family of elem N-struct. on U_α, $\{U_\alpha\}$ open cover of M, with certain compatibility conditions on intersections.

Thm: (CFG, 92) $\forall \varepsilon > 0$ $\exists\, \delta_n, \rho_n > 0$, if (M,g) $|\sec_g| \leq 1$, $\mathrm{injrad}_g \leq \delta_n$, $\exists\, g_\varepsilon$; $\exists$ N-struct

- $e^{-\varepsilon} g \leq g_\varepsilon \leq e^{+\varepsilon} g$
- g_ε is "invariant" for ~~the~~ the N-structure on M.
- This N-structure is defined using open sets U_α, and $\forall x \in M$, $B(x, \rho_n) \subset U_\alpha$ for some α.

Rmk: → N-structures of thm live at macroscopic scale, contrarily to F-struct.

Properties of these N-structures:

1) The thm implies that FM is union of "N-orbits" of form $\Gamma \backslash N$. Quotient by $O(n)$, these N-orbits project to orbits inside M, each orbit is $\approx$ almost flat mfd.

2) Recall F-structures defined using torus actions. Similarly here around each point of FM, $\exists$ $\hat{U}$, $\exists$ $\tilde{\hat{U}} \to \hat{U}$ on which an nilpotent N is acting with discrete kernel.

3) For N as above, $\mathrm{center}(N) \neq 0$ and $\Gamma \backslash \mathrm{center}(N) \simeq$ torus T. The project orbits of N-structure contains the projections of those T, get flat mfd T' which is of >0 dimension.

These T' are orbits of an F-struct of $\underline{>0 \text{ rank}}$. (N-structure of $\underset{\text{Thm}}{}$)

4) $\forall\, x \in (M, g_\varepsilon)$ is contained in a neighborhood of the form

$\approx$ R - tubular neighb of Q

where Q = an orbit in M of the N-structure
= diffeo to finite quotient of a nilmfd $\Gamma \backslash N$

where $R = R(n, \varepsilon) > 0$.

IV. Collapsing construction bis & other application

N-structures of CFG induce a canonical F-structure of >0 rank.
N-structures of CFG are "uniformly controlled" so, the canonical F-structures are also "uniform. controlled".

Improved collapsing construction:

By applying CG collapsing construction to that canonical F-structure, one gets:

$\forall (M,g)$ $|sec_g| \leq 1$, injrad $< \delta_n$, g A-regular, then

$\exists \{g_t\}_{t \in (0,1]}$, $g_1 = g$,

- $\text{injrad}_{g_t} \longrightarrow 0$ at a rate only depending on (n, A).
- $|sec_{g_t}| \leq C(n, A)$

Rmk: there are local version of that improved collaps.

Other applications:

Recall MinVol, ess-MinVol, recall that $\forall M$,

$\exists (M_\infty, g_\infty) \in \overline{\mathcal{M}}^{weak}_{|sec| \leq 1}(M)$, $\quad ess MinVol(M) = Vol(M_\infty, g_\infty)$

M_∞ is obtained by M by removing collapsed regions.

a) $\forall M$, M_∞ as above, $\#$ conn comp of $M_\infty < \infty$
(apply local version of improved coll const)

b) $\inf\{\mathrm{Vol}(M,g') \,;\, \sec_{g'} \geq -1\} =: V_{\sec\geq -1}$

$< \underline{\text{ess-MinVol}} \quad (\leq \mathrm{MinVol})$

(apply a variation of the local collapsing const).

c) Recall BCG: $\forall X_{hyp}$ hyp mfd,
$\inf\{\mathrm{Vol}(M,g'') \,;\, \mathrm{Ric}_{g''} \geq -(n-1)\}$
$= \mathrm{Vol}(X_{hyp}, hyp)$

$\Rightarrow \mathrm{MinVol}(X_{hyp}) = \mathrm{essMinVol}(X_{hyp}) =$ hyperbolic vol.

d) in dim 2 or 3 $\mathrm{essMinVol} = \mathrm{MinVol} = V_{\sec\geq -1}$.

Question: Many definitions for (ess) MinVol make sense for $\mathrm{Ric} \geq -(n-1)$.

One can try to "essentialise" $\inf \{ \mathrm{Vol} \mid \mathrm{Ric} \geq -(n-1) \}$

(replace $\mathrm{injrad} > \delta$ by $\mathrm{Vol}(B(x,1)) > \delta$)

Can one prove such results for this ess-minvol ...?