An introduction to Brakke flows

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Contents

1 Preface

These are lecture notes for a mini course given in June 2021 at the Summer School 2021: Curvature Constraints and Spaces of Metrics at the Institut Fourier in Grenoble.

We omit many fundamental and important results on smooth mean curvature flow but focus on an introduction to one of the weak notions of mean curvature flow, known as Brakke flows. We have freely copied from the lectures notes

- B. White, *Topics in mean curvature flow*, lecture notes by O. Chodosh. Available at http://web.stanford.edu/~ochodosh/notes.html

of the beautiful course by Brian White, with some further simplifications to adjust to the format. So we claim in no way originality.

Here is a list of further introductory texts on mean curvature flow:

- K. Ecker, *Regularity Theory for Mean Curvature Flow*, Birkhäuser

- C. Mantegazza, *Lecture Notes in Mean Curvature Flow*, Progress in Mathematics, Volume 290, Birkhäuser

- R. Haslhofer, *Lectures on mean curvature flow.* Available at https://arxiv.org/abs/1406.7765.

- R. Haslhofer, *Lectures on mean curvature flow of surfaces.* Available at https://arxiv.org/abs/2105.10485

2 Geometry of Hypersurfaces

We give an introduction to the geometry of hypersurfaces in Euclidean space. For a more detailed background, we recommend [9, Chapter 6] and [19, §7].

We restrict ourselves to manifolds of codimension 1 in an Euclidean ambient space, i.e. we consider a n-dimensional smooth manifold M, without boundary, either closed or complete and non-compact and an immersion (or embedding)

$$F : M \to \mathbb{R}^{n+1}.$$

We call the image $F(M)$ a hypersurface. We will often identify points on M with their image under the immersion, if there is no risk of confusion.

Let $x = (x_1, \ldots, x_n)$ be a local coordinate system on M. The components of a vector v in the given coordinate system are denoted by v^i, the ones of a covector w are w_i. Mixed tensors have components with upper and lower indices depending on their type. We denote by

$$g_{ij} = \left\langle \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} \right\rangle_e$$

the induced metric on M, where $\langle \cdot, \cdot \rangle_e$ is the Euclidean scalar product on $\mathbb{R}^{n+1}$. Note that the metric g induces anatural isomorphism between the tangent and the cotangent space. In coordinates, this is expressed in terms of raising/lowering indexes by means of the matrcies g_{ij} and g^{ij}, where g^{ij} is the inverse of g_{ij}. The scalar product on the tangent bundle naturally extends to any tensor bundle. For instance the scalar product of two $(1,2)$-tensors T^i_{jk} and S^i_{jk} is defined by

$$\langle T^i_{jk}, S^i_{jk} \rangle = T^{jk}_i S^i_{jk} = T^l_{pq} S^i_{jk} g_{li} g^{pj} g^{qk} \, .$$

The norm of a tensor T is then given by $|T| = \sqrt{\langle T, T \rangle}$. The volume element $d\mu$ (which is just the restriction of the n-dimensional Hausdorff measure to M), is given in local coordinates by

$$d\mu = \sqrt{\det g_{ij}} \, dx$$

Recall that on the ambient space $\mathbb{R}^{n+1}$ we have the standard covariant derivative $\bar{\nabla}$ given via directional derivatives of each coordinate, i.e. for two smooth vectorfields on

X, Y on $\mathbb{R}^{n+1}$ we have

$$\bar{\nabla}_X Y\Big|_p = (D_{X(p)}Y^1(p), \cdots, D_{X(p)}Y^{n+1}(p))$$

where $Y(p) = (Y^1(p), \cdots, Y^{n+1}(p))$, and $D_{X(p)}$ is the directional derivative at p in direction $X(p)$. Recall that to define $D_{X(p)}Y^i(p)$ it is only necessary to locally know Y along an integral curve to X through p. Given two vectorfields V, W along $F(M)$ and tangent to M we thus define the connection

$$\nabla_V W := (\bar{\nabla}_V W)^T ,$$

where ^{T} is the projection to the tangent space of M. One can check that this is the Levi-Civita connection corresponding to the induced metric g. In coordinates we obtain for the derivative of a vector v^i or a covector w^i the formulas

$$\nabla_k v^i = \frac{\partial v^i}{\partial x_k} + \Gamma^i_{jk} v^j, \qquad \nabla_k w_j = \frac{\partial w_j}{\partial x_k} - \Gamma^i_{jk} w_i,$$

where Γ^i_{jk} are the Christoffel symbols of the the connection ∇. This covariant derivative extends to tensors of all kind, in coordinates, we have e.g. for a (1,2)-tensor T^i_{jl}:

$$\nabla_k T^i_{jl} = \frac{\partial T^i_{jl}}{\partial x_k} + \Gamma^i_{mk} T^m_{jl} - \Gamma^m_{jk} T^i_{ml} - \Gamma^m_{kl} T^i_{jm} \ ,.$$

If f is a function, we set $\nabla_k f = \frac{\partial f}{\partial x_k}$, which concides with the differential $df\Big(\frac{\partial}{\partial x_k}\Big)$. Using the isomorphism induced by the metric g we can regard ∇f also as element of the tangent space, in this case it is called the *gradient* of f. The gradient of f can be identified with a vector in $\mathbb{R}^{n+1}$ via the differential dF; such a vector is called the *tangential gradient* of f and is denoted by $\nabla^M f$, given in coordinates by

$$\nabla^M f = \nabla^i f \frac{\partial F}{\partial x_i} = g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial F}{\partial x_i} \ .$$

The word "tangential" comes from the equivalent definition of $\nabla^M f$ in case f is a function defined on the ambient space $\mathbb{R}^{n+1}$. It can be checked that $\nabla^M f$ is the projection of the standard Euclidean gradient DF onto the tangent space of M, that is

$$\nabla^M f = Df - \langle Df, \nu \rangle_e \, \nu$$

where ν is a local choice of unit normal to M.

For two tangential vectorfields V, W, the shape operator is given by

$$S_V W = (\bar{\nabla}_V W)^{\perp}$$

where $^{\perp}$ is the projection to the normal space of M. Thus we have

$$\bar{\nabla}_V W = \nabla_V W + S_V W \ .$$

For local choice of unit normal vector field ν, the second fundamental form of M, a $(0,2)$-tensor, is given by

$$A(V,W) = -\langle S_V W, \nu \rangle_e = \langle W, \bar{\nabla}_V \nu \rangle_e \ ,$$

or in coordinates $A = (h_{ij})$ by

$$h_{ij} = -\left\langle \frac{\partial^2 F}{\partial x_j \partial x_i}, \nu \right\rangle_e = \left\langle \frac{\partial F}{\partial x_i}, \frac{\partial}{\partial x_j} \nu \right\rangle_e \ .$$

The matrix of the Weingarten map $W(X) = \bar{\nabla}_X \nu : T_pM \to T_pM$ is given by $h^i_{\ j} = g^{il} h_{lj}$. The *principal curvatures* of M at a point are the eigenvalues of the symmetric matrix h^i_j, or equivalently the eigenvalues of h_{ij} with respect to g_{ij}. We denote the principal curvatures by $\lambda_1 \leq \cdots \leq \lambda_n$. The *mean curvature* is defined as the trace of the second fundamental form, i.e.

$$H = h^i_{\ i} = g^{ij} h_{ij} = \lambda_1 + \ldots + \lambda_n \ .$$

The square of the norm of the second fundamental form will be denoted by

$$|A|^2 = g^{mn} g^{st} h_{ms} h_{nt} = h^n_{\ s} h^s_{\ n} = \lambda_1^2 + \ldots + \lambda_n^2 .$$

It is easy to see that $|A|^2 \geq H^2/n$, with equality only if all the curvatures coincide; in fact we have the identity

$$|A|^2 - \frac{1}{n} H^2 = \frac{1}{n} \sum_{i<j} (\lambda_i - \lambda_j)^2 \ . \tag{2.1}$$

Clearly, A, W, H depend on the choice of orientation; if ν is reversed, their sign changes. But note that the *mean curvature vector*

$$\mathbf{H} = -H\nu$$

is independent of the orientation; in particular it is well defined globally even if M is non-orientable.

We will call a hypersurface *convex* if the principal curvatures are non-negative everywhere. Observe that, with these definitions, if $F(M)$ is the boundary of a convex set, and the normal is outward pointing, then all principal curvatures are non-negative.

Recall the curvature tensor

$$R(X,Y,Z,W) = g(\nabla_X \nabla_Y W - \nabla_Y \nabla_X W - \nabla_{[X,Y]}W, Z)$$

for vectorfields X, Y, Z, W on M.The Gauss equations relate the Riemann w.r.t. g to the curvature tensor of the ambient space in terms of the second fundamental form. Since the Euclidean ambient space is flat, we obtain

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk} \, .$$

Thus the scalar curvature is given by

$$R = g^{ik}g^{jl}R_{ijkl} = H^2 - |A|^2 = 2\sum_{i<j} \lambda_i \lambda_j \ .$$

We also recall the *Codazzi equations*, which say that

$$\nabla_i h_{jk} = \nabla_j h_{ik} \, , \qquad i, j, k \in \{1, \dots, n\},$$

i.e. taking into account the symmetry of h_{ij}, this implies that the tensor $\nabla A = \nabla_i h_{jk}$ is totally symmetric.

Let $X \in C^1_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$, i.e. an ambient vectorfield with compact support. Let $(\phi_t)_{-\varepsilon<t<\varepsilon}$ be the associated family of diffeomorphisms, i.e.

$$\frac{\partial \phi_t}{\partial t} = X(\phi_t) \, , \qquad \phi_0 = \mathrm{id} \ .$$

We then obtain a one-parameter family of variations of $F(M)$ via $\phi_t(F(M)$. We compute the variation of the measure as

$$\begin{aligned} \frac{\partial d\mu}{\partial t}\Big|_{t=0} &= \frac{\partial \sqrt{\det g_{ij}}}{\partial t}\Big|_{t=0} dx = \frac{1}{\sqrt{\det g_{ij}}} (\det g_{ij}) g^{rs} \left\langle \frac{\partial X}{\partial x_r}, \frac{\partial F}{\partial x_s} \right\rangle_e dx \\ &= g^{rs} \left\langle \bar\nabla_{\frac{\partial F}{\partial x_r}} X, \frac{\partial F}{\partial x_s} \right\rangle_e d\mu \, , \end{aligned} \tag{2.2}$$

which leads us to define the tangential divergence

$$\mathrm{div}_M X = g^{ij} \left\langle \bar\nabla_{\frac{\partial F}{\partial x_i}} X, \frac{\partial F}{\partial x_j} \right\rangle_e = \sum_{i=1}^n \langle \bar\nabla_{e_i} X, e_i \rangle_e$$

where $e_1, \cdots, e_n$ is an ON-basis of T_pM. Recall the divergence theorem on a closed manifold

$$\int_M \mathrm{div}_M(X)\, d\mu = 0\,, \tag{2.3}$$

for $X \in \mathrm{Vec}_c(M)$. This follows directly from Stokes' theorem. For the normal part of a non-tangential vector field, one obtains

$$\begin{aligned}\mathrm{div}_M(X^\perp) &= \mathrm{div}_M(\langle X, \nu\rangle_e\, \nu) = \langle \nabla^M \langle X, \nu\rangle_e, \nu\rangle_e + \langle X, \nu\rangle_e \mathrm{div}_M \nu \\ &= \langle X, \nu\rangle_e g^{ij} \left\langle \bar{\nabla}_{\frac{\partial F}{\partial x_i}} \nu, \frac{\partial F}{\partial x_j} \right\rangle_e = \langle X, \nu\rangle_e g^{ij} h_{ij} = \langle X, \nu\rangle_e H = -\langle X, \mathbf{H}\rangle_e\end{aligned}$$

Together with (2.3) this yields the general divergence theorem

$$\int_M \mathrm{div}_M(X)\, d\mu = \int_M \mathrm{div}_M(X^T) + \mathrm{div}_M(X^\perp)\, d\mu = -\int_M \langle X, \mathbf{H}\rangle_e\, d\mu\,, \tag{2.4}$$

for $X \in \mathrm{Vec}_c(\mathbb{R}^{n+1})$. Together with (2.2) this yields the first variation formula

$$\frac{\partial}{\partial t}\bigg|_{t=0} \int_{\phi_t(M)} 1\, d\mu_t = \int_M \mathrm{div}_M(X)\, d\mu = -\int_M \langle X, \mathbf{H}\rangle_e\, d\mu\,. \tag{2.5}$$

We recall the *Laplace-Beltrami operator* on functions $f : M \to \mathbb{R}$ given by

$$\Delta^M f = \mathrm{div}_M(\nabla^M f)\ .$$

We write simply Δ instead of Δ^M. One can easily check that

$$\Delta^M f = g^{ij}\nabla_i\nabla_j f = g^{ij}\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma^k_{ij}\frac{\partial f}{\partial x_k}\right) = \frac{1}{\sqrt{\det g_{ij}}}\frac{\partial}{\partial x_i}\left(\sqrt{\det g_{ij}}\, g^{ij}\frac{\partial f}{\partial x_j}\right)\ .$$

The divergence theorem then gives the usual integration by parts formula

$$\int_M f\Delta h\, d\mu = -\int_M \langle \nabla f, \nabla h\rangle\, d\mu = \int_M h\Delta f\, d\mu\ .$$

If f is a function on the ambient space we have by the above calculations

$$\begin{aligned}\Delta^M f &= \mathrm{div}_M(\nabla^M f) = \mathrm{div}_M(Df) - \mathrm{div}_M(Df^\perp) \\ &= \Delta^{\mathbb{R}^{n+1}} f - D^2 f(\nu, \nu) + \langle Df, \mathbf{H}\rangle_e\ .\end{aligned} \tag{2.6}$$

Thus Δ^M not only neglects the contribution of the second derivatives normal to M, but also takes into account the curvature of M.

Let $X = (x_1, \ldots, x_{n+1})$ be the coordinates of $\mathbb{R}^{n+1}$. Equation (2.6) yields

$$\Delta^M x_i = \langle \mathbf{H}, e_i\rangle_e$$

where e_i is the i-th basis vector of $\mathbb{R}^{n+1}$. We can thus write

$$\Delta^M X = \mathbf{H} \ .$$

Note that in coordinates the vectorfield X is just given by F, and we can write

$$\Delta^M F = \mathbf{H} \ .$$

We also note the identity

$$\Delta^M |X|_e^2 = 2n + 2\langle X, \mathbf{H}\rangle_e \,. \tag{2.7}$$

The second fundamental form corresponds in a certain sense to second derivatives of an immersion, and its symmetry reflects that second partial derivatives of a function commute. Similarly the Codazzi equations can be seen as a geometric manifestation that third partial derivatives commute. Thus we can also expect that there is a symmetry of the second covariant derivatives of the second fundamental form. This identity is known as *Simon's identity*:

$$\nabla_k \nabla_l h_{ij} = \nabla_i \nabla_j h_{kl} + h_{kl} h_i{}^m h_{mj} - h_{km} h_{il} h^m_j + h_{kj} h_i{}^m h_{ml} - h_k{}^m h_{ij} h_{ml} \tag{2.8}$$

For a proof see [15]. We note the following two consequences

$$\Delta h_{ij} = \nabla_i \nabla_j H + H h_i{}^m h_{mj} - h_{ij}|A|^2 \tag{2.9}$$

and

$$\frac{1}{2}\Delta |A|^2 = h^{ij} \nabla_i \nabla_j H + |\nabla A|^2 + H \mathrm{tr}(A^3) - |A|^4 \ . \tag{2.10}$$

We give the explicit expressions of the main geometric quantities in the case when $F(M)$ is the graph of a function $x_{n+1} = u(x_1, \dots, x_n)$. We choose the orientation where ν points downwards. By straightforward computations one gets

$$\nu = \frac{(D_1 u, \dots, D_n u, -1)}{\sqrt{1 + |Du|^2}} \ , \tag{2.11}$$

$$g_{ij} = \delta_{ij} + D_i u D_j u, \qquad g^{ij} = \delta_{ij} - \frac{D_i u D_j u}{1 + |Du|^2} \,, \tag{2.12}$$

$$h_{ij} = \frac{D^2_{ij} u}{\sqrt{1 + |Du|^2}}, \qquad H = \mathrm{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) \ , \tag{2.13}$$

where div is the standard divergence on $\mathbb{R}^n$.

We also recall the strong maximum principle scalar functions:

Theorem 2.1 (Strong maximum principle for parabolic equations)**.** *Let* (M, g_t) *be a closed Riemannian manifold with a smooth family of metrics* $(g_t)_{t\in[0,T)}$ *and* $f : M \times [0,T) \to \mathbb{R}$ *satisfying*

$$\frac{\partial f}{\partial t} \geq \Delta f + b^i \nabla_i f + c\, f$$

for some smooth funtions b^i, c*, where* $c \geq 0$*. If* $f(\cdot, 0) \geq 0$ *then*

$$\min_M f(\cdot, t) \geq \min_M f(\cdot, 0)\,.$$

Furthermore, if $f(p, t_0) = \min_M f(\cdot, 0)$ *for some* $p \in M,\ t > 0$*, then* $f \equiv \min_M f(\cdot, 0)$ *for* $0 \leq t \leq t_0$*.*

For a proof see for example [10, Chapter 6.4 and Chapter 7.1.4] .

3 Basic properties

Let M^n be closed (or non-compact and complete), and $F : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a smooth family of immersions. Let $M_t := F(M,t)$. We call this family a mean curvature flow starting at an initial immersion F_0, if

$$
\begin{aligned}
\frac{\partial F}{\partial t} &= -H \cdot \nu = \mathbf{H} \quad (= \Delta_{M_t} F) \\
F(\cdot, 0) &= F_0 \ .
\end{aligned}
\tag{3.1}
$$

Remark 3.1: i) In general, it suffices to ask that

$$\left(\frac{\partial F}{\partial t}\right)^{\perp} = \mathbf{H}\,.$$

One solves the ODE on M given by

$$\frac{\partial \phi}{\partial t} = -dF^{-1}\left(\left(\frac{\partial F}{\partial t}\right)^{T}\right)(\phi)$$

with $\phi(0) = \mathrm{id}$. Then $\tilde{F} := F \circ \phi$ solves usual MCF.

ii) The evolution equation for a surface, which is locally given as the graph of a function u, is thus

$$\left(\frac{\partial u}{\partial t}\, e_{n+1}\right)^{\perp} = \mathbf{H}$$

or equivalently

$$\frac{\partial u}{\partial t}\,\langle e_{n+1}, \nu\rangle = -H\ ,$$

which yields

$$\frac{\partial u}{\partial t} = \sqrt{1+|Du|^2}\ \mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \left(\delta^{ij} - \frac{D^i u\, D^j u}{1+|Du|^2}\right) D_i D_j u\,. \tag{3.2}$$

This is a quasilinear parabolic equation.

iii) By formula (2.4) we have for an evolution with normal speed $-f\nu$ that

$$\frac{d}{dt}|M_t| = \frac{d}{dt}\int_M 1\, d\mu_t = -\int_M fH\, d\mu\,,$$

and thus for mean curvature flow

$$\frac{d}{dt}|M_t| = -\int_M |\mathbf{H}|^2\, d\mu_t\,.$$

By the Hölder's inequality, mean curvature flow decreases area the fastest, when comparing with speeds with the same L^2-norm. Furthermore, along mean curvature flow one has the natural estimate

$$\int_0^T\int |\mathbf{H}|^2\, d\mu_t\, dt = |M_0| - |M_T| \leq |M_0|\,.$$

Examples: There are not many explicit examples of mean curvature flow solutions.

i) The most basic one is the evolution of a sphere with initial radius $R > 0$. Assuming that the solutions remains rotationally symmetric (which follows from uniqueness, see later), we obtain the following ODE for the radius $r(t)$:

$$\frac{\partial r}{\partial t} = -\frac{n}{r}.$$

with initial condition $r(0) = R$. Integrating yields $r(t) = \sqrt{R^2 - 2nt}$. Note that the maximal existence time $T = R^2/(2n)$ is finite and the curvature blows up for $t \to T$. Furthermore, the shrinking sphere is an example of a solution which only moves by scaling, a so-called *self-similar shrinker*.

By the previous example the evolution of a cylinder

$$\mathbb{S}^k_R \times \mathbb{R}^{n-k}$$

remains cylindrical with radius given by $r(t) = \sqrt{R^2 - 2kt}$. Note that again this solution is *self-similarly shrinking*.

Another class of examples are translating solutions. Assuming that they translate with speed one in direction τ, they satisfy the elliptic equation

$$H = -\langle \tau, \nu\rangle.$$

Assuming that the solution is graphical, i.e. $x_{n+1} = u(x_1, \cdots, x_n)$, and moving in e_{n+1} direction we obtain from (3.2) that it satisfies the equation

$$\left(\delta^{ij} - \frac{D^i u\, D^j u}{1+|Du|^2}\right) D_i D_j u = 1\,.$$

In one dimension the equation becomes

$$y_{xx} = 1 + y_x^2$$

which can be integrated explicitly, yielding $y(x) = -\ln\cos x$ for $|x| < \pi/2$, up to translation and adding constants. This solution is usually called the *grim reaper*.
In higher dimensions it can be shown that there is a unique, convex, rotationally symmetric solution - but which is defined on the whole space. For properties of this solution see [7]. For $n = 2$ these are the unique convex translating entire graphs, but for $n \geq 3$ there exist entire convex translating graphs which are not rotationally symmetric, see [20].

The upwards translating grim reaper given by $e^{-y(t)} = e^{-t}\cos x(t)$ and the downwards translating grim reaper given by $e^{y(t)} = e^{-t}\cos x(t)$ can be combined to give another pair of solutions given implicitly as the solution set of

$$\cosh y(t) = e^t \cos x(t)\ , \tag{3.3}$$

and

$$\sinh y(t) = e^t \cos x(t)\ . \tag{3.4}$$

The *paperclip*, given as solution of (3.3) restricted to $|x| < \pi/2$ desribes a compact *ancient* solution that for $t \to 0$ becomes extinct in a round point and for $t \to -\infty$ looks like two copies of the grim reaper glued together smoothly. The *hairclip* (3.4) is an *eternal* solution, which for $t \to -\infty$ looks like an infinite row of grim reapers, alternating between translating up and translating down, and for $t \to +\infty$ converges to a horizontal line.

We have the following short-time existence result.

Theorem 3.2 (Short-time existence). *Let $F_0 : M^n \to \mathbb{R}^{n+1}$ be a smooth immersion of* *a closed n-dimensional manifold M. Then there exists a unique smooth solution on a maximal time interval $[0,T)$ for $T \in (0,\infty]$.*

The difficulty to prove this result comes from the geometric nature of the flow, which makes any solution invariant under diffeomorphisms of M and thus the evolution equation is only weakly parabolic. There different ways to prove this result. One can either follow the approach of Hamilton [13] for the Ricci flow and use the Nash-Moser Implicit function theorem. Alternatively one can use the so-called De Turck trick to break the diffeomeorphism invariance. The maybe most natural way for mean curvature flow is to write the evolving surfaces $M_t = F(M,t)$ for a short time as an exponential normal graph over $M_0 = F_0(M)$. One can then check that the height function u satisfies a quasilinear parabolic equation similar to (3.2) for which standard results for those type of equations can be applied. For details see [15].

The strong maximum principle implies the following.

Theorem 3.3 (Avoidance principle)**.** *Assume two solutions to mean curvature flow $(M_t^1)_{t\in[0,T)}$ and $(M_t^2)_{t\in[0,T)}$ are initially disjoint (and at least one of them is compact), i.e. $M_0^1 \cap M_0^2 = \emptyset$. Then $M_t^1 \cap M_t^2 = \emptyset \;\; \forall\, t \in (0,T)$.*

Proof. Assume that this is not the case. Then there exists a first time $t_0 \in (0,T)$ where $M_{t_0}^1$ and $M_{t_0}^2$ touch at the point $x_0 \in \mathbb{R}^{n+1}$. Note that this implies that $T_{x_0}M_{t_1}^1 = T_{x_0}M_{t_1}^2 := T$ and there is an $\varepsilon > 0$ such that we can write $(M_t^1)_{t_0-\varepsilon\le t\le t_0}$ and $(M_t^2)_{t_0-\varepsilon\le t\le t_0}$ locally as graphs over the affine space $x_0 + T$. The two graph functions u_1, u_2 satisfy (3.2) which we write as

$$\frac{\partial u}{\partial t} = \left(\delta^{ij} - \frac{D^i u\, D^j u}{1+|Du|^2}\right) D_{ij}u =: a^{ij}(Du) D_{ij}u.$$

We can assume w.l.o.g that $u_2 \le u_1$ and $u_1 = u_2$ at (x_0,t_0). But note that $v = u_1 - u_2$

satisfies a linear parabolic equation:

$$\begin{aligned}\frac{\partial v}{\partial t} &= a^{ij}(Du_1)D_iD_ju_1 - a^{ij}(Du_2)D_iD_ju_2\\ &= \int_0^1 \frac{d}{ds}\big(a^{ij}(D(su_1+(1-s)u_2)D_{ij}(su_1+(1-s)u_2)\big)\,ds\\ &= \left(\int_0^1 a^{ij}(D(su_1+(1-s)u_2))\,ds\right)D_{ij}v\\ &\quad + \left(\int_0^1 \frac{\partial a^{ij}}{\partial p_k}(D(su_1+(1-s)u_2))D_{ij}(su_1+(1-s)u_2)\,ds\right)D^kv\\ &=: \tilde{a}^{ij}D_{ij}v + \tilde{b}^kD_kv\,,\end{aligned}$$

where p is the Du variable of $a^{ij}(p)$. Note that $\tilde{a}^{ij}$ is symmetric and strictly positve. Since $v \geq 0$ and $v = 0$ at (x_0, t_0) the strong maximum principle implies that $v \equiv 0$ which yields a contradiction. □

With more or less the same argument one can show the following.

Corollary 3.4 (Preservation of embeddedness)**.** *If M_0 is closed and embedded, then M_t is embedded for all t.*

Remark 3.5: (i) Enclosing a compact initial hypersurface M_0 by a large sphere, and using that the maximal existence time of the evolution of the sphere is finite, we obtain that the maximal existence time T is finite.

(ii) Note the we can translate a solution to mean curvature flow in the ambient space and get a new solution to mean curvature flow. Thus the avoidance principle implies that the distance between two disjoint solutions is non-decreasing in time.

(iii) In case M_0 is embedded, we will always choose ν to be the *outward* unit normal.

4 Weak compactness for submanifolds

To understand compactness for mean curvature flow, we see that the evolution equation for the measure gives a natural bound on the space-time integral of H^2. We are thus naturally led to trying to take limits of submanifolds under some weak curvature bounds. To be concrete, suppose that M_i is a sequence of m-submanifolds in $\mathbb{R}^N$. Assume that the areas of M_i are locally uniformly bounded. We may later assume that

$$\int_{M_i} |\mathbf{H}|^2 \, d\mu_i$$

is also locally uniformly bounded, but this will not be important in the beginning. We ask if it is possible to understand a weak limit of the M_i.

The simplest possibility is as follows: note that any m-submanifold M determines a Radon measure μ_M by

$$\mu_M(S) = \mathcal{H}^m(M \cap S)\,.$$

Equivalently, for any compactly supported continuous function f, we set

$$\int f \, d\mu_M := \int_M f \, d\mathcal{H}^m\,.$$

Hence, for the M_i as before, we may pass to a subsequence so that $\mu_{M_i} \rightharpoonup \mu$ weakly.

This is quite a coarse procedure (as we will see later), and we would like a more refined definition. An important observation is that M actually defines a Radon measure on $G(m,N)$, where $G(m,N)$ is the Grassmanian of m-dimensional subspaces in $\mathbb{R}^N$. We define the measure V_M:

$$\int f \, dV_M = \int_M f(x, Tan(M,x)) \, d\mathcal{H}^m$$

for $f : \mathbb{R}^N \times G(m,N) \to \mathbb{R}$ a continuous function of compact support, where $Tan(M,x)$

is the tangent space of M at x. Alternatively, we have

$$V_M(S) = \mathcal{H}^m(\{x \in M : (x, Tan(M,x)) \in S\}).$$

We can take a subsequence so that $V_{M_i} \rightharpoonup V$. Note that for $\pi : \mathbb{R}^N \times G(m,N) \to U$ the natural projection map, we can check that $\pi_* V_M = \mu_M$.

Definition 4.1. *An m-dimensional* varifold *in $\mathbb{R}^N$, is a Radon measure on $\mathbb{R}^N \times G(m,N)$.*

4.1 Examples of varifold convergence

See the handwritten notes from class.

4.2 Integral varifolds

The class of general varifolds will be way to general and will include numerous pathological examples. We thus would like to define a smaller class.

Lemma 4.2. *Suppose that M, M' are m-dimensional C^1-submanifolds of $\mathbb{R}^N$. Let*

$$Z = \{x \in M \cap M' : Tan(M,x) \neq Tan(M',x)\}.$$

Then, $\mathcal{H}^m(Z) = 0$.

Proof. If M, M' are hypersurfaces, then Z is an $(m-1)$-dimensional C^1-submanifold by transversality. In higher co-dimension, one may show that $M \cap M'$ is contained in a $(m-1)$-dimensional C^1-submanifold, by projecting to a lower dimensional space. □

Corollary 4.3. *Suppose that $S \subset \cup_i M_i$ and $S \subset \cup_i M_i'$ for M_i, M_i', m-dimensional*

C^1-submanifolds of $\mathbb{R}^N$. Define T on S by

$$T(x) = Tan(M_i, x)\,,$$

where i is the first i so that $x \in M_i$, i.e., $x \in M_i \setminus \cup_{j<i} M_j$. Define T' similarly. Then

$$T(x) = T'(x)$$

for a.e. $x \in S$.

Thus, for a S a Borel subset of a C^1-submanifold $M \subset \mathbb{R}^N$, we define V_S to be the varifold given by

$$\int f\, dV_S = \int_S f(x, \mathrm{Tan}(M, x))\, d\mathcal{H}^m\,.$$

This does not depend on the choice of M, by the previous corollary. This allows us to give the following two equivalent definitions.

Definition 4.4. *An* integral m-varifold *is a varifold which can be written as*

$$V = \sum_{i=1}^{\infty} V_{S_i}\,.$$

Definition 4.5. *Suppose that $\theta \in \mathscr{L}^1_{loc}(U; \mathbb{Z}^+; \mathcal{H}^m)$ and $S = \{\theta > 0\} \subset Z \cup (\cup_i M_i)$ where Z has $\mathcal{H}^m(Z) = 0$ and the M_i are m-dimensional C^1-submanifolds of U. This data defines an* integral m-varifold V_θ *by*

$$\int f\, dV_\theta = \int_S f(x, T(x))\theta(x)\, d\mathcal{H}^m(x) = \sum_i \int_{M_i \setminus \cup_{j<i} M_j} f(x, \mathrm{Tan}(M_i, x))\theta(x)\, d\mathcal{H}^m\,.$$

Note to relate the two definitions, we can easily see that $\theta = \sum \mathbf{1}_{S_i}$.

4.3 First variation of a varifold

First we recall

Theorem 4.6 (Divergence theorem)**.** *Suppose that M is a C^2-submanifold of $\mathbb{R}^N$ and X a compactly suppported C^1-vectorfield. We set*

$$\mathrm{div}_M X = \sum_i \langle \nabla_{e_i} X, e_i \rangle \,,$$

for $e_1, \cdots, e_m$ an orthonormal basis for $\mathrm{Tan}(M,x)$. Then

$$\begin{aligned} \int_M \mathrm{div}_M X &= \int_M \mathrm{div}_M X^\perp + \int_M \mathrm{div}_M X^T \\ &= -\int_M \langle X, \mathbf{H} \rangle \, d\mathcal{H}^m + \int_{\partial M} \langle X, \mathbf{n} \rangle \, d\mathcal{H}^{m-1} \,, \end{aligned}$$

where $\mathbf{n}$ is the exterior unit conormal to ∂M.

Note that the left hand side makes sense if M is just C^1. If there is a distributional vector field H making this true, then we say that H is the *weak mean curvature.*

Now, for V an m-varifold, we define the *first variation* of V by

$$\delta V(X) = \int \mathrm{div}_T X \, dV(x,T) \,,$$

where

$$\mathrm{div}_T X = \sum_i \langle \nabla_{e_i} X, e_i \rangle$$

for $e_1, \cdots, e_m$ an orthonormal basis for T. If V is an integral varifold, this can be written as

$$\delta V(X) = \int \mathrm{div}_{T(V,x)} X \, d\mu_V \,.$$

Remark 4.7: If X is a C^1-vectorfield with support in $K \subset U$, K compact, and $(\phi_t)_{-\varepsilon<t<\varepsilon}$ the associated 1-parameter family of diffeomorphims with $\phi_0 = \mathbf{id}$, then one can show (see for example L. Simon's lecture notes, [19]), that

$$\left.\frac{d}{dt}\right|_{t=0} \big(((\phi_t)_\# V)(K)\big) = \delta V(X)$$

as in the smooth case. (This just follows from expanding the Jacobian).

Note that trivially, if $V_i \rightharpoonup V$, then $\delta V_i(X) \to \delta V(X)$. Suppose now that we have local bounds on the first variation in the form

$$|\delta V(X)| \leq C_K \|X\|_0$$

for supp$X \subset K \Subset \mathbb{R}^N$. Then the Riesz represantation theorem implies that there is a Radon measure λ on $\mathbb{R}^N$ and a λ-measurable unit vectorfield Λ such that

$$\delta V(X) = \int \langle X, \Lambda \rangle \, d\lambda \, .$$

Decomposing λ with respect to μ_V, there is $\lambda_{\mathrm{ac}} \ll \mu_V$ and λ_{sing} so that

$$\begin{aligned} \delta V(X) &= \int \langle X, \Lambda \rangle \, d\lambda_{\mathrm{ac}} + \int \langle X, \Lambda \rangle \, d\lambda_{\mathrm{sing}} \\ &= \int \left\langle X, \underbrace{\Lambda \frac{d\lambda_{\mathrm{ac}}}{d\mu_V}}_{=:-\mathbf{H}} \right\rangle d\mu_V + \int \langle X, \underbrace{\Lambda}_{=:\mathbf{n}} \rangle \, d\lambda_{\mathrm{sing}} \end{aligned}$$

Thus, in the case that V has locally bounded first variation, we have

$$\delta V(X) = -\int \langle X, \mathbf{H} \rangle \, d\mu_V + \int \langle X, \mathbf{n} \rangle \, d\lambda_{\mathrm{sing}} \, .$$

The following deep theorem due to Allard [1] is the reason reason that the class of integral varifolds is a reasonable one to study.

Theorem 4.8 (Allard's compactness theorem)**.** *Suppose that $V_i \rightharpoonup V$ is a sequence of integral varifolds converging weakly to a varifold V. If the V_i have locally uniformly bounded first variation, i.e., for $K \Subset \mathbb{R}^N$ there is C_K independent of i such that for all C^1-vectorfields X with supp $X \subset K$, we have*

$$|\delta V_i(X)| \leq C_K \|X\|_0$$

then V is also an integral varifold.

Note that in the theorem, we trivially obtain the bounds

$$|\delta V(X)| \leq C_K \|X\|_0$$

For example, a sequence of hypersurfaces satisfy the hypothesis of Allard's theorem if and only if

$$\int_K |\mathbf{H}_i|\, d\mu_{M_i} + \int_K d\sigma_i \leq C_K\,,$$

where $d\sigma_i$ is the boundary measure for M_i.

Example 4.9: Note that the quantities "$|\mathbf{H}|$" and "$d\sigma$" can be "mixed up" in the limit. For example consider a sequence of ellipses converging to a line with multiplicity two. Note that $\int |\mathbf{H}_i| = 2\pi$ and $\sigma_i = 0$, but in the limit, $\mathbf{H} = 0$ but $\sigma \neq 0$. Conversely, a sequence of polygons converging to a circle has $\mathbf{H} = 0$, but nontrivial boundary measure (at the vertices), but the circle only has mean curvature and no boundary.

Theorem 4.10. *Suppose that for V_i integral varifolds, we have $V_i \rightharpoonup V$ and that V_i has locally bounded first variation and no generalized bounday. Equivalently, we are assuming that*

$$\delta V_i(X) = -\int \langle X, \mathbf{H}_i\rangle\, d\mu_i\,.$$

Assume that

$$\int_K |\mathbf{H}_i|^2\, d\mu_i \leq C_K < \infty$$

for $K \Subset U$. Then

(1) V is an integral varifold.

(2) We have

$$\int \langle \mathbf{H}_i, X\rangle\, d\mu_i \to \int\int \langle \mathbf{H}, X\rangle\, d\mu\,,$$

where X is a continuous vectorfield with compact support in U.

We note that any L^p for $p > 1$ could replace L^2 here.

Proof. We first note that $V_i \rightharpoonup V$ implies local mass bounds. Thus, local bounds for $\mathbf{H}_i$ in L^2 imply local bounds in L^1, and V is an integral varifold, by Allard's theorem. Note that

$$\delta V_i(X) = -\int_K \langle \mathbf{H}_i, X\rangle \leq \left(\int_K |\mathbf{H}_i|^2\right)^{\frac{1}{2}} \left(\int_K |X|^2\right)^{\frac{1}{2}} \leq C_K^{\frac{1}{2}} \left(\int_K |X|^2\right)^{\frac{1}{2}}\,.$$

Thus,

$$|\delta V(X)| \leq C_K^{\frac{1}{2}} \left(\int_K |X|^2 \right)^{\frac{1}{2}} .$$

From this, it follows from the Riesz representation theorem that $\mathbf{H} \in \mathscr{L}^2_{\text{loc}}(d\mu_V)$. Since we thus have local uniform bounds in L^2 for $\mathbf{H}_i$ and $\mathbf{H}$, for the statement in (2) we can approximate any continuous vectorfield with compact support on U by a C^1-vectorfield with compact support on U. The stated convergence for a C^1-vectorfield then follows by the definition of of the first variation and the convergence $V_i \rightharpoonup V$. □

5 Brakke flow

We now discuss Brakke's weak mean curvature flow [2, 18], we follow here the conventions used in [21].

Definition 5.1. *An (n-dimensional)* integral Brakke flow *in* $\mathbb{R}^{n+1}$ *is a* 1*-parameter family of Radon measures* $(\mu(t))_{t\in I}$ *over an interval* $I \subset \mathbb{R}$ *so that:*

1. *For almost every* $t \in I$*, there exists an integral* n*-dimensional varifold* $V(t)$ *with* $\mu(t) = \mu_{V(t)}$ *so that* $V(t)$ *has locally bounded first variation and has mean curvature* $\mathbf{H}$ *orthogonal to* $Tan(V(t), \cdot)$ *almost everywhere.*

2. *For a bounded interval* $[t_1, t_2] \subset I$ *and any compact set* $K \subset \mathbb{R}^{n+1}$*,*

$$\int_{t_1}^{t_2} \int_K (1 + |\mathbf{H}|^2) d\mu(t) dt < \infty.$$

3. *If* $[t_1, t_2] \subset I$ *and* $f \in C^1_c(\mathbb{R}^{n+1} \times [t_1, t_2])$ *has* $f \geq 0$ *then*

$$\int f(\cdot, t_2)\, d\mu(t_2) - \int f(\cdot, t_1)\, d\mu(t_1) \leq \int_{t_1}^{t_2} \int \left(-|\mathbf{H}|^2 f + \langle \mathbf{H}, \nabla f\rangle + \tfrac{\partial}{\partial t} f\right)\, d\mu(t)\, dt.$$

We will often write $\mathcal{M}$ *for a Brakke flow* $(\mu(t))_{t\in I}$*, with the understanding that we're referring to the family* $I \ni t \mapsto \mu(t)$ *of measures satisfying Brakke's inequality.*

Remark 5.2: We note that if M_t is a smooth mean curvature flow, then

$$\begin{aligned} \frac{d}{dt} \int_{M_t} f\, dA &= \int_{M_t} \left(-\langle \mathbf{H}, v\rangle f + \langle \nabla^{\perp} f, v\rangle + \frac{\partial f}{\partial t} \right) dA \\ &= \int_{M_t} \left(-|\mathbf{H}|^2 f + \langle \nabla f, \mathbf{H}\rangle + \frac{\partial f}{\partial t} \right) dA \end{aligned}$$

where the first equality holds for any smooth flow with velocity v. An obvious question is why we require the inequality, rather than the equality in the definition of Brakke flow. The reason for this is that only the inequality is possibly preserved under limits. For example, the weak limit of rescaled grim reapers is a multiplicity two line for $t < 0$ and is empty for $t > 0$!

Theorem 5.3. *Suppose that (μ_t) is an n-dimensional integral Brakke flow on $\mathbb{R}^{n+1}$. Let $\phi = (r^2 - |x|^2 - 2nt)^+$. Then*

$$\int \phi^4 \, d\mu_t$$

is decreasing in t.

Proof. For $f = \frac{1}{4}\phi^4$, we compute

$$\nabla f = \phi^3 \nabla \phi \,,$$

so

$$\mathrm{div}_M(\nabla f) = 3\phi^2 |\nabla^T \phi|^2 + \phi^3 \mathrm{div}_M(\nabla \phi) \geq -2n\phi^3 \,.$$

Moreover,

$$\frac{\partial f}{\partial t} = \phi^3 \frac{\partial \phi}{\partial t} = -2n\phi^3 \,.$$

Thus, using f as a test function in the definition of Brakke flow yields

$$\begin{aligned} \int f \, d\mu_b - \int f \, d\mu_a &\leq \int_a^b \int \left(-|\mathbf{H}|^2 + \langle \mathbf{H}, \nabla f \rangle + \frac{\partial f}{\partial t} \right) d\mu_t \, dt \\ &\leq \int_a^b \int \left(-\mathrm{div}_M(\nabla f) + \frac{\partial f}{\partial t} \right) d\mu_t \, dt \\ &\leq 0 \,, \end{aligned}$$

as desired. □

Corollary 5.4. *For an integral n-dimensional Brakke flow μ_t on $\mathbb{R}^{n+1}$ defined on $[a, b]$, for K compact, we have uniform mass bounds, i.e., there is c_K independent of t, so that*

$$\mu_t(K) \leq c_K < \infty$$

for $t \in [a,b]$.

Theorem 5.5. *An integral Brakke flow satisfies*

$$\int_a^b \int_K |\mathbf{H}|^2 \, d\mu_t \, dt < C_K(1+b-a)$$

for any $K \Subset \mathbb{R}^{n+1}$.

Proof. Suppose that $\phi \in C_c^2, \phi \geq 0$, is time independent. Recall that $\frac{|\nabla \phi|^2}{\phi} \leq C(|\nabla^2 \phi|)$ (Exercise). Hence, since

$$\langle \nabla \phi, \mathbf{H} \rangle \leq \frac{1}{2} \frac{|\nabla \phi|^2}{\phi} + \frac{1}{2} \phi |\mathbf{H}|^2 \,,$$

we have

$$\begin{aligned}
\int \phi \, d\mu_a - \int \phi \, d\mu_b &\geq \int_a^b \int \left(\phi H^2 - \langle \nabla \phi, \mathbf{H} \rangle \right) \, d\mu_t \, dt \\
&\geq \int_a^b \int \left(\frac{1}{2} \phi |\mathbf{H}|^2 - \frac{1}{2} \frac{|\nabla \phi|^2}{\phi} \right) \, d\mu_t \, dt
\end{aligned}$$

Rearranging, we obtain

$$\begin{aligned}
\int_a^b \int \frac{1}{2} \phi |\mathbf{H}|^2 \, d\mu_t \, dt &\leq \int \phi \, d\mu_a - \int \phi \, d\mu_b + \frac{1}{2} \int_a^b \int \frac{|\nabla \phi|^2}{\phi} \, d\mu_t \, dt \\
&\leq \int \phi \, d\mu_a - \int \phi \, d\mu_b + C(\phi) \int_a^b \int \chi_{\{\phi \neq 0\}} \, d\mu_t \, dt \\
&\leq C(\phi) c_K (1+b-a) \,,
\end{aligned}$$

where $\{\phi \neq 0\} \subset K$. ☐

Theorem 5.6. *An integral n-dimensional Brakke flow satisfies*

$$\lim_{\tau \nearrow t} \mu_\tau \geq \mu_t \geq \lim_{\tau \searrow t} \mu_\tau \,.$$

In other worhds, for $\phi \in C_c^0(\mathbb{R}^{n+1})$ *with* $\phi \geq 0$*, we have*

$$\lim_{\tau \nearrow t} \int \phi \, d\mu_\tau \geq \int \phi \, d\mu_t \geq \lim_{\tau \searrow t} \int \phi \, d\mu_\tau \, .$$

Proof. First, assume that $\phi \in C_c^2(\mathbb{R}^{n+1})$ with $\phi \geq 0$ (the general case follows by approximation). Then, we have

$$\begin{aligned} \int \phi \, d\mu_d - \int \phi \, d\mu_c &\leq \int_c^d \int \left(-\phi|\mathbf{H}|^2 + \langle \mathbf{H}, \nabla\phi \rangle\right) \, d\mu_t \, dt \\ &\leq \int_c^d \int \frac{1}{2} \frac{|\nabla \phi|^2}{\phi} \, d\mu_t \, dt \\ &\leq C(\phi) c_{\operatorname{supp}\phi}(d-c) \, . \end{aligned}$$

Thus

$$f(t) := \int \phi \, d\mu_t - C(\phi) c_{\operatorname{supp}\phi} t$$

is decreasing in t. This implies that

$$f(t^-) \geq f(t) \geq f(t^+) \, ,$$

which finishes the proof, as the linear part of f is continuous. □

Note that we have shown

Theorem 5.7. *For an inegral* n*-dimensional Brakke flow on* $\mathbb{R}^{n+1}$ *and* $\phi \in C_c^2(\mathbb{R}^{n+1}), \phi \geq 0$*, the map*

$$t \mapsto \int \phi \, d\mu_t - C(\phi) c_{supp\,\phi} t$$

is decreasing.

5.1 A compactness theorem for integral Brakke flows

Theorem 5.8. *Suppose that $[a, b] \ni t \mapsto \mu_t^i$ is a sequence of integral Brakke flows. Assume that the local bounds on area are uniform, i.e.*

$$\sup_i \sup_{t \in [a,b]} \mu_t^i(K) \le c_K < \infty$$

for all $K \Subset \mathbb{R}^{n+1}$. Then after passing to a subsequence

(1) we have weak convergence $\mu_t^i \rightharpoonup \mu_t$ for all $t \in [a, b]$,

(2) $[a, b] \ni t \mapsto \mu_t$ is an integral Brakke flow,

(3) for a.e. $t \in [a, b]$, after passing to a further subsequence which depends on t, the associated varifolds converge nicely $V_t^i \to V_t$.

Proof. Choose $\phi \in C_c^2(\mathbb{R}^{n+1}), \phi \ge 0$. Recall that

$$L_i^{\phi}(t) = \int \phi \, d\mu_t^i - c(\phi) c_{\operatorname{supp} \phi} t$$

is a sequence of uniformly bounded, decreasing functions of t. Passing to a subsequence depending ϕ, we have that $L_i^{\phi}(t)$ converges pointwise to a decreasing functin $L(t)$. Hence,

$$\int \phi \, d\mu_t^i$$

has a limit, for all $t \in [a, b]$. Now choose a countable, dense subset $\mathcal{S} \subset C_c^0(\mathbb{R}^{n+1}; \mathbb{R}^+)$ of functions in $C_c^2(\mathbb{R}^{n+1}; \mathbb{R}^+)$. Repeating the above process for each $\phi \in \mathcal{S}$ (choosing a diagonal subsequence), we see that there is a subsequence in i such that for all $\phi \in \mathcal{S}$,

$$\int \phi \, d\mu_t^i$$

has a limit, for all $t \in [a, b]$. By density this extends to all of $C_c^0(\mathbb{R}^{n+1}; \mathbb{R}^+)$. Since the limits are unique, we have that

$$\mu_t^i \rightharpoonup \mu_t$$

for a family of Radon measures $[a, b] \ni t \mapsto \mu_t$. We now want to show that this is a Brakke flow and prove the stated strengthened convergence.

Now, we replace $\mathbb{R}^{n+1}$ by $U \Subset \mathbb{R}^{n+1}$, for simplicity. Thus we may assume that $\mu_t^i(U) \leq C < \infty$ independent of i and t. Note that we have also shown earlier that we can thus assume that

$$\int_a^b \int |\mathbf{H}|^2 d\mu_t^i \, dt \leq D < \infty$$

independent of i. Let $[c,d] \subset [a,b]$. Then

$$\int \phi \, d\mu_c^i - \int \phi \, d\mu_d^i \geq \int_c^d \int \left(\phi |\mathbf{H}_i|^2 - \langle \nabla\phi, \mathbf{H}_i \rangle - \frac{\partial \phi}{\partial t} \right) d\mu_t^i \, dt \, .$$

Thus,

$$\begin{aligned} &\int \phi \, d\mu_c^i - \int \phi \, d\mu_d^i + \varepsilon D + \int_c^d \int \frac{1}{2\varepsilon} |\nabla\phi|^2 \, d\mu_t^i \, dt \\ &\quad \geq \int_c^d \int \left(\phi |\mathbf{H}_i|^2 - \langle \nabla\phi, \mathbf{H}_i \rangle + \varepsilon |\mathbf{H}_i|^2 + \frac{1}{2\varepsilon} |\nabla\phi|^2 - \frac{\partial \phi}{\partial t} \right) d\mu_t^i \, dt \end{aligned}$$

Note that

$$\langle \nabla\phi, \mathbf{H}_i \rangle \leq \frac{1}{2\varepsilon} |\nabla\phi|^2 + \frac{\varepsilon}{2} |\mathbf{H}_i|^2 \, ,$$

so

$$\phi |\mathbf{H}_i|^2 - \langle \nabla\phi, \mathbf{H}_i \rangle + \varepsilon |\mathbf{H}_i|^2 + \frac{1}{2\varepsilon} |\nabla\phi|^2 \geq \frac{\varepsilon}{2} |\mathbf{H}_i|^2 \, ,$$

which in particular is positive. Now, we may pass to the limit in i and use Fatou's lemma to see that

$$\begin{aligned} &\int \phi \, d\mu_c - \int \phi \, d\mu_d + \varepsilon D + \int_c^d \int \frac{1}{2\varepsilon} |\nabla\phi|^2 \, d\mu_t \, dt \\ &\quad \geq \int_c^d \liminf_i \int \left(\phi |\mathbf{H}_i|^2 - \langle \nabla\phi, \mathbf{H}_i \rangle + \varepsilon |\mathbf{H}_i|^2 + \frac{1}{2\varepsilon} |\nabla\phi|^2 \right) d\mu_t \, dt - \int_c^d \int \frac{\partial \phi}{\partial t} \, d\mu_t \, dt \, . \end{aligned}$$

Thus, for a.e. $t \in [c,d]$ we have

$$C(t) := \liminf_i \int \left(\phi |\mathbf{H}_i|^2 - \langle \nabla\phi, \mathbf{H}_i \rangle + \varepsilon |\mathbf{H}_i|^2 + \frac{1}{2\varepsilon} |\nabla\phi|^2 \right) d\mu_t < \infty \, .$$

Pass to a subsequence (depending on t!) so that this becomes a limit rather than a lim inf. Because the integrand is bounded from below by $\frac{\varepsilon}{2}|\mathbf{H}_i|^2$, we see that the μ_t^i are integral varifolds with mean curvature uniformly in $\mathscr{L}^2(\mu_t^i)$. Hence, we can apply the strengthened form of Allard's compactness theorem to pass to a subsequence so that

$$V_t^i \rightharpoonup V_t\,,$$

where V_t is an integral varifold with $\mathbf{H} \in \mathscr{L}^2(d\mu_V)$. In particular

$$\int \langle \mathbf{H}_i, X\rangle \, d\mu_{V_t^i} \to \int \langle \mathbf{H}, X\rangle \, d\mu_{V_t}$$

for $X \in C_c(U; \mathbb{R}^{n+1})$ a continuous vector field.

Note that for a.e. t, $V(t)$ is well defined independent of the subsequence depending on t. This is because an integral varifold V is uniquely determined by its associate measure μ_V. However, we emphasise that the convergence of V_t^i to V as varifolds requires extracting a subsequence depending on t, as we have done above.

Now, returning to $C(t)$, each term converges to what we expect, except for the $|\mathbf{H}_i|^2$ terms, which might drop in general (by weak convergence (just use duality)). Hence we see that

$$C(t) \geq \int \left(\phi|\mathbf{H}|^2 - \langle \nabla\phi, \mathbf{H}\rangle + \varepsilon|\mathbf{H}|^2 + \frac{1}{2\varepsilon}|\nabla\phi|^2 \right) d\mu_t\,.$$

Cancelling the terms with $\frac{1}{2\varepsilon}|\nabla\phi|^2$ and letting $\varepsilon \to 0$, this goes in the right direction to conclude that μ_t is an integral Brakke flow. □

5.2 Existence by elliptic regularisation

We now describe Ilmanen's construction [18] of Brakke flows by "elliptic regularisation". For the technical details see [18].

Theorem 5.9. *Let z denote the height function in $\mathbb{R}^{n+2} = \mathbb{R}^{n+1} \times \mathbb{R}$ and $\vec{e}$ the upward pointing unit vector. Then $M \subset \mathbb{R}^{n+2}$ is a critical point for $\int_M e^{-\lambda z}\, dA$ if and only if*

$$t \mapsto M - \lambda t \vec{e}$$

is a mean curvature flow.

Proof. If $s \mapsto M_s$ is a variation of M with velocity X, we compute

$$\frac{d}{ds}\int_{M_s} e^{-\lambda z}\, dA = \int \langle -\mathbf{H} + \nabla^\perp(-\lambda z), X\rangle e^{-\lambda z}\, dA = \int \langle -\mathbf{H} - \lambda \vec{e}^\perp, X\rangle e^{-\lambda z}\, dA\,.$$

On the other hand, note that the flow $t \mapsto M - \lambda t\vec{e}$ has velocity $-\lambda\vec{e}$ and thus normal velocity $-\lambda\vec{e}^\perp$. Comparing these two computations proves the theorem. □

Now, for Σ a compact n-dimensional surface in $\mathbb{R}^{n+1}$, let $M_\lambda \subset \mathbb{R}^{n+2}$ minimize $\int e^{-\lambda z}\, dA$ subject to the constraint $\partial M_\lambda = \Sigma$. We can show that M_λ exists and in nice situations (e.g, for hypersurfaces of low dimensions) is regular except for a small singular set. So by the above computation $t \mapsto M_\lambda - \lambda t\vec{e}$ is a mean curvature flow. (This also works in the general case, it then yields a translating Brakke flow).

Our goal is to send $\lambda \to +\infty$. We would like to show that these converge to a limit Brakke flow μ_t which is translation invariant, i.e. $\mu_t = \Sigma_t \times \mathbb{R}$ for Σ_t an n-dimensional Brakke flow in $\mathbb{R}^{n+1}$ with $\Sigma_0 = \Sigma$.

Set $M_\lambda(a, b) = M_\lambda \cap \{a < z < b\}$ and set $S_\lambda(z_0) = M_\lambda \cap \{z = z_0\}$ (see figure in the handwritten notes). Then, we have that, for ν the upward pointing normal vector to $\partial M_\lambda(a, b)$ in $M_\lambda(a, b)$

$$\begin{aligned} 0 &= \int_{M_\lambda(a,b)} \operatorname{div}_M(\vec{e}) = \int_{M_\lambda(a,b)} -\langle \mathbf{H}, \vec{e}\rangle + \int_{S_\lambda(b)} \langle \vec{e}, \nu\rangle - \int_{S_\lambda(a)} \langle \vec{e}, \nu\rangle \\ &= \int_{M_\lambda(a,b)} \lambda|\vec{e}^\perp|^2 + \int_{S_\lambda(b)} |\vec{e}^T| - \int_{S_\lambda(a)} |\vec{e}^T|\,. \end{aligned}$$

We may rearrange this to yield

$$\int_{M_\lambda(a,b)} \lambda|\vec{e}^\perp|^2 + \int_{S_\lambda(b)} |\vec{e}^T| = \int_{S_\lambda(a)} |\vec{e}^T|\,.$$

In particular

$$z \mapsto \int_{S_\lambda(z)} |\vec{e}^T|$$

is a decreasing function. Now, we have

$$\begin{aligned}\operatorname{area}(M_\lambda(a,b)) &= \int_{M_\lambda(a,b)} |\vec{e}^\perp|^2 + |\vec{e}^T|^2 \\ &\leq \frac{1}{\lambda}\int_{S_\lambda(0)} |\vec{e}^T| + \int_{M_\lambda(a,b)} |\vec{e}^T|^2 \\ &= \frac{1}{\lambda}\int_{S_\lambda(0)} |\vec{e}^T| + \int_{z=a}^{z=b}\int_{S_\lambda(z)} |\vec{e}^T| \\ &\leq (\lambda^{-1} + b - a)\int_{S_\lambda(0)} |\vec{e}^T| \\ &\leq (\lambda^{-1} + b - a)\operatorname{area}(\Sigma)\,.\end{aligned}$$

Thus, the flows have uniform area bounds on compact sets in space-time. Thus, a subsequence converges to a limit Brakke flow (strictly speaking, these flows have boundary, but we could work in the upper half-space, i.e. $\{z > 0\}$, where they do not have boundary).

We have thus obtianed a Brakke flow $\mathbb{R}^+ \ni t \mapsto \mu_t$ in $\mathbb{R}^{n+1} \times \mathbb{R}^+$. We would like to show that (1) the flow is translation invariant, i.e. $\mu_t = \Sigma_t \times \mathbb{R}^+$ for Σ_t a Brakke flow in $\mathbb{R}^{n+1}$ and (2) the has initial condition $\Sigma \times \mathbb{R}^+$.

We start showing the translation invariance. Suppose ϕ is a nice compactly supported nonnegative function on $\mathbb{R}^{n+1} \times \mathbb{R}^+$. Define $\phi^\tau(x,z) = \phi(x, z-\tau)$ to be "upward translation by τ". Let $t \mapsto \mu_t^\lambda$ be the Brakke flow constructed above, which limits to μ_t along some subsequence $\lambda \to \infty$.

Note that (we will use the shorthand $\nu(f) = \int f\, d\nu$ for ν a Radon measure)

$$\mu_t^\lambda(\phi^\tau) = \mu_{t+\tau/\lambda}^\lambda(\phi)\,.$$

Recall that there is a constant c_ϕ depending on ϕ, but independent of λ so that

$$t \mapsto \mu_t^\lambda(\phi) - c_\phi t$$

is decreasing in time. Hence, if $t < s$, for λ large so that

$$t < t + \tau/\lambda < s\,,$$

we see that

$$\mu_t^\lambda(\phi) - c_\phi t \geq \mu_{t+\tau/\lambda}^\lambda(\phi) - c_\phi(t+\tau/\lambda) = \mu_t(\phi^\tau) - c_\phi(t+\tau/\lambda) \geq \mu_s^\lambda(\phi) - c_\phi s\,.$$

Sending $\lambda \to \infty$ along the subsequence so that $\{\mu_t^\lambda\}$ converges to $\{\mu_t\}$, we have that

$$\mu_t(\phi) - c_\phi t \geq \mu_t(\phi^\tau) - c_\phi t \geq \mu_s(\phi) - c_\phi s\,,$$

i.e. we have

$$\mu_t(\phi) \geq \mu_t(\phi^\tau) \geq \mu_s(\phi) - c_\phi(s-t)\,.$$

Sending $s \nearrow t$, we have

$$\mu_t(\phi) \geq \mu_t(\phi^\tau) \geq \mu_{t^+}(\phi)\,.$$

This holds for every t. Moreover, for all but countably many t, μ_t is continuous at t. Thus, we see that for a.e. t, $\mu_t = \Sigma_t \times \mathbb{R}^+$, as desired.

Now, we would to show that $\mu_0 = \mu_{\Sigma\times\mathbb{R}^+}$. To do so, we will use the flat norm $\mathscr{F}(\cdot)$. For A, B closed m-dimensional cycles, the flat norm $\mathscr{F}(A-B)$ is the infimum of the area of $m+1$ chains spanning $A-B$.

Let π denote the projection onto $\mathbb{R}^n \times \{b\}$ and let $A_\lambda = \pi(M_\lambda(0,b))$. We compute by the area formula

$$\text{area}(A_\lambda) \leq \int_{M_\lambda(0,b)} |\vec{e}^\perp| \leq \left(\int_{M_\lambda(0,b)} |\vec{e}^\perp|^2\right)^{\frac{1}{2}} (\text{area}(M_\lambda(0,b)))^{\frac{1}{2}}\,.$$

The area term is uniformly bounded. Moreover, we have seen above that the divergence theorem implies that

$$\int_{M_\lambda(0,b)} |\vec{e}^\perp|^2 \leq \lambda^{-1}\text{area}(\Sigma)\,.$$

Putting this together, we see that area$(A_\lambda) \to 0$. Because b is bounded, we can then use this to see that the area in the blue region in the figure (see the handwritten notes from class) is tending to zero.

This shows that

$$\mathscr{F}(M_\lambda(0,b) + A_\lambda - \Sigma \times [0,b]) \to 0\,.$$

Recall that the mass is lower semicontinuous under flat convergence. In particular, we have that

$$\begin{aligned}\operatorname{area}(\Sigma\times[0,b]) &\le \liminf_{\lambda\to\infty}\operatorname{area}(M_\lambda(0,b)+A_\lambda)\\ &\le \liminf_{\lambda\to\infty}\operatorname{area}(M_\lambda(0,b))\\ &\le \limsup_{\lambda\to\infty}\operatorname{area}(M_\lambda(0,b))\\ &\le \limsup_{\lambda\to\infty}\,(\lambda^{-1}+b)\operatorname{area}(\Sigma)\\ &= b\operatorname{area}(\Sigma)\\ &= \operatorname{area}(\Sigma\times(0,b))\,.\end{aligned}$$

In particular, in addition to flat norm convergence, the masses converge (rather than dropping down)! We may now use the following general result:

Proposition 5.10. *Suppose $T_i \to T$ in flat norm. Then, we have seen that the masses satisfy $\mathbb{M}(T) \le \liminf \mathbb{M}(T_i)$. By passing to a subsequence, we may assume that the associated Radon measures μ_{T_i} converge. Then, we have that*

$$\mu_T \le \liminf_{i\to\infty} \mu_{T_i}\,.$$

Roughly speaking, this means that even locally mass can only drop down (we already used the global version of this fact). So, if the total mass converges, then the measures must converge (if they dropped down somewhere, then because the mass cannot jump up somewhere else, this would mean that the mass actually dropped down).

This combines to show that $\mu_0 = \Sigma \times \mathbb{R}^+$. Thus, we have completed the existence theory. Note that we have found solutions Σ_t with the extra convenient property that the flow $t \mapsto \Sigma_t \times \mathbb{R}^+$ is the limit of smooth (or with a small singular set) flows.

5.3 Monotonicity formula

In this section we will first establish that Huisken's monotonicity formula also holds for Brakke flows and use it to show that one always has the existence of self-similarly shrinking tangent flows. In the following we will always assume that our initial measure of the Brakke flow has bounded n-dimensional area ratios, that is

$$\sup_{x\in\mathbb{R}^{n+1}} \sup_{r>0} \frac{\mu_0(B_r(x))}{\omega_n r^n} \le D < \infty \tag{5.1}$$

where ω_n is the measure of the unit ball in $\mathbb{R}^n$.

Lemma 5.11. *Assume $(\mu_t)_{t\ge 0}$ is an n-dimensional integral Brakke flow on $\mathbb{R}^{n+1}$ satisfying (5.1). Then for $t \in [0, r^2/(4n)]$*

$$\sup_{x\in\mathbb{R}^{n+1}} \mu_t(B_r(x)) \le 2^{n+2} D\, r^n\,.$$

Proof. This follows from (5.1) together with Theorem 5.3. Exercise. □

We recall that for $X_0 = (x_0, t_0)$ a point in space time, we consider the (scaled) backwards heat kernel

$$\rho_{X_0}(x,t) := (4\pi(t_0 - t))^{-n/2} e^{-\frac{|x-x_0|^2}{4(t_0-t)}}\,.$$

Proposition 5.12. *Let $(\mu_t)_{t\ge 0}$ be an n-dimensional integral Brakke flow on $\mathbb{R}^{n+1}$ satisfying (5.1). Then for $X_0 = (x_0, t_0)$ with $t_0 > 0$ we have for all $0 \le t_1 < t_2 < t_0$,*

$$\int \rho_{X_0}(\cdot, t_2)\, d\mu_{t_2} + \int_{t_1}^{t_2}\int \Big|\mathbf{H} + \frac{(x-x_0)^\perp}{2(t_0-t)}\Big|^2 \rho_{X_0}\, d\mu_t\, dt \le \int \rho_{X_0}(\cdot, t_1)\, d\mu_{t_1}\,.$$

Proof. By a translation in space we can assume $x_0 = 0$. We denote $\rho = \rho_{(0,t_0)}$. We recall that from the definition of Brakke flow, we have

$$\int \phi(\cdot, t_2)\, d\mu_{t_2} - \int \phi(\cdot, t_1)\, d\mu_{t_1} \le \int_{t_1}^{t_2}\int \left(-|\mathbf{H}|^2\phi + \langle \mathbf{H}, \nabla\phi\rangle + \frac{\partial\phi}{\partial t}\right) d\mu_t\, dt\,,$$

for $\phi \in C_c^1(\mathbb{R}^{n+1}\times\mathbb{R};\mathbb{R}^+)$. Whenever the inner integral is finite (i.e. μ_t comes from an n-dimensional varifold with first variation in $\mathscr{L}^2_{\mathrm{loc}}(\mu_t)$), we can calculate, using the first variation formula

$$\begin{aligned}\int -|\mathbf{H}|^2\phi + \langle \mathbf{H}, \nabla\phi\rangle + \frac{\partial\phi}{\partial t}\, d\mu_t &= \int -|\mathbf{H}|^2\phi + 2\langle \mathbf{H}, \nabla\phi\rangle + \mathrm{div}_{\mathrm{Tan}(V_{\mu_t})}(\nabla\phi) + \frac{\partial\phi}{\partial t}\, d\mu_t\\ &= \int -\phi\Big|\mathbf{H} - \frac{\nabla^\perp\phi}{\phi}\Big|^2 + Q_{\mathrm{Tan}(V_{\mu_t})}(\phi)\, d\mu_t\end{aligned}$$

where for any n-dimensional subspace T of $\mathbb{R}^{n+1}$

$$Q_T(\phi) = \frac{|\nabla^\perp \phi|^2}{\phi} + \mathrm{div}_T(\nabla\phi) + \frac{\partial \phi}{\partial t}\,.$$

Note that $Q_T(\rho) = 0$.

To insert ρ into the above formula, let $\psi = \psi_R$ be a cutoff function with $\chi_{B_R(0)} \le \psi \le \chi_{B_{2R}(0)}$, $R|\nabla\psi| + R^2|\nabla^2\psi| \le C$. We calculate

$$Q_T(\psi\rho) = \psi\, Q_T(\rho) + \rho\, Q_T(\psi) + 2\langle \nabla\psi, \nabla\rho\rangle \le C\left(\frac{1}{R^2} + \frac{1}{t_0 - t}\right)\chi_{B_{2R}(0)\setminus B_R(0)}\,\rho\,,$$

where we used the fact that $|\nabla\rho| \le \rho|x|/(2(t_0-t))$. Inserting $\psi\rho$ above, we obtain

$$\begin{aligned}\int \psi\rho\, d\mu_{t_2} &+ \int_{t_1}^{t_2}\int \psi\left|\mathbf{H} + \frac{x^\perp}{2(t_0-t)} - \frac{\nabla^\perp\psi}{\psi}\right|^2 \rho\, d\mu_t\, dt \\ &\le \int \psi\rho\, d\mu_{t_1} + \left(\frac{C}{R^2} + \frac{C}{t_0 - t_2}\right)\int_{t_1}^{t_2}\int_{B_{2R}(0)\setminus B_R(0)} \rho\, d\mu_t\, dt\end{aligned}$$

Note that Lemma 5.11 implies that

$$\sup_{t_1 \le t \le t_2} \int \rho\, d\mu_t < \infty\,.$$

Thus the result follows by the monotone and dominated convergence theorems. □

5.3.1 Entropy

For $M^n \subset \mathbb{R}^{n+1}$, we define

$$F(M) = \frac{1}{(4\pi)^{\frac{n}{2}}}\int_M e^{-\frac{|x|^2}{4}}\, d\mathcal{H}^n\,,$$

or more generally for a Radon measure μ, we set

$$F_n(\mu) = \frac{1}{(4\pi)^{\frac{n}{2}}}\int_M e^{-\frac{|x|^2}{4}}\, d\mu\,.$$

We set $A(r) = \mu(B(0,r))$. Note that $A(r)$ is increasing in r and thus $A'(r)$ exists as a radon measure on $\mathbb{R}$. Then we have by integration by parts (assuming that $A(r)$ grows sub-exponentially) that

$$F_n(\mu) = \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^\infty e^{-\frac{r^2}{4}} A'(r)\, dr = \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^\infty \frac{\omega_n}{2} r^{n+1} e^{-\frac{r^2}{4}} \left(\frac{A(r)}{\omega_n r^n}\right) dr\,.$$

Note that we can also estimate for any $r_0 > 0$

$$\begin{aligned} F_n(\mu) \geq & \frac{1}{(4\pi)^{\frac{n}{2}}} \int_{r_0}^\infty \frac{r}{2} e^{-\frac{r^2}{4}} A(r)\, dr = -\frac{1}{(4\pi)^{\frac{n}{2}}} \int_{r_0}^\infty A(r) \frac{d}{dr}\left(e^{-\frac{r^2}{4}}\right) dr \\ = & \frac{1}{(4\pi)^{\frac{n}{2}}} A(r_0) e^{-\frac{r_0^2}{4}} \end{aligned}$$

In particular we see that $F_n(\mu)$ controls the area ratios and is controlled by their supremum, i.e.

$$\frac{\omega_n r^n}{(4\pi)^{\frac{n}{2}}} e^{-\frac{r^2}{4}} \left(\frac{A(r)}{\omega_n r^n}\right) \leq F_n(\mu) \leq C \sup_{r \geq 1} \frac{A(r)}{\omega_n r^n}\,.$$

Colding and Minicozzi have defined [8] a related quantity, *entropy*, as

$$\lambda(M) = \sup_{\lambda > 0, p \in \mathbb{R}^{n+1}} F(\lambda M + p)\,,$$

where we define $\lambda_n(\mu)$ correspondingly. By the above bounds, we see that there exists $C = C(n) > 0$ such that for $A(p,r) = \mu(B_r(p))$,

$$C^{-1} \sup_{\substack{p \in \mathbb{R}^{n+1},\\ r > 0}} \frac{A(p,r)}{\omega_n r^n} \leq \lambda_n(\mu) \leq C \sup_{\substack{p \in \mathbb{R}^{n+1},\\ r > 0}} \frac{A(p,r)}{\omega_n r^n}\,. \tag{5.2}$$

From the monotonicity formula for Brakke flows we obtain:

Corollary 5.13. *Let $(\mu_t)_{t \geq 0}$ be an n-dimensional integral Brakke flow on $\mathbb{R}^{n+1}$ satisfying (5.1). Then the entropy $\lambda_n(\mu_t)$ is finite and decreasing with respect to t. Furthermore, the area ratios area uniformly controlled for all time.*

Note that, in comparison, Lemma 5.11 gives control on the area ratios for small $r > 0$ only for short time (with $t \to 0$ as $r \to 0$). The monotonicity formula allows to rule out measure concentration for all times.

5.3.2 Tangent flows

For a given m-dimensional, integral Brakke flow $(\mu_t)_{t\in I}$ on $\mathbb{R}^{n+1}$ and $\lambda > 0$, we denote the parabolically rescaled measures at a point $X_0 = (x_0, t_0)$ by

$$\mu_t^{X_0,\lambda}(A) = \lambda^n \mu_{t_0+\lambda^{-2}t}(\lambda^{-1}\cdot A + x_0) \tag{5.3}$$

for $t \in I_{\lambda,t_0} := \lambda^2(I - t_0)$. It is easy to check that $I_{\lambda,t_0} \ni t \mapsto \mu_t^{X_0,\lambda}$ is again an n-dimensional, integral Brakke flow on $\mathbb{R}^{n+1}$. Furthermore, if the initial flow has entropy bounded by C, then so does $(\mu_t^{X_0,\lambda})$.

Proposition 5.14 (Existence of tangent flows)**.** *For a given n-dimensional, integral Brakke flow $(\mu_t)_{t\in[0,\infty}$ on $\mathbb{R}^{n+1}$ satisfying (5.1), and any point $X_0 = (x_0, t_0)$ with $t_0 >$ 0 and any sequence $\lambda_i \to \infty$, there exists a subsequence (labelled the same) and a Brakke flow $(\nu_t)_{t\in\mathbb{R}}$, such that $(\mu_t^{X_0,\lambda_i}) \rightharpoonup (\nu_t)$ (with convergence as guaranteed by the compactness theorem for Brakke flows), and*

$$\nu_t(A) = \nu_t^\lambda(A) := \lambda^n \nu_{\lambda^{-2}t}(\lambda^{-1}\cdot A), \qquad t < 0 \tag{5.4}$$

for all $\lambda > 0$, and ν_{-1} satisfies

$$H + \frac{x^\perp}{2} = 0 \qquad \nu_{-1}\text{-}a.e.\, x\,. \tag{5.5}$$

Furthermore

$$\int \rho_{(0,0)}(\cdot, t)\, d\nu_t = \lim_{t'\nearrow t_0} \int \rho_{X_0}\, d\mu_{t'} \qquad t < 0\,. \tag{5.6}$$

Proof. We write $\mu_t^\lambda = \mu_t^{X_0,\lambda}$ and $\rho_{(0,0)} = \rho$. By (5.1) and Corollary 5.13 the flows (μ_t^λ) have bounded area ratios, independent of λ (Exercise). By the compactness theorem for Brakke flows, there exists a subsequence (labelled the same) $\lambda_i \to \infty$ such that $(\mu_t^{\lambda_i}) \rightharpoonup (\nu_t)$ for all $t \in \mathbb{R}$. Since the flows have uniformly bounded area ratios, for every $t < 0$ and $\varepsilon > 0$ there exists $R > 0$ such that

$$\sup_i \int_{\mathbb{R}^{n+1}\setminus B_R(0)} \rho\, d\mu_t^{\lambda_i} \le \varepsilon\,.$$

Using a suitable cutoff function, weak convergence $\mu_t^{\lambda_i} \rightharpoonup \nu_t$ implies that

$$\int \rho(\cdot,t)\,d\nu_t = \lim_{i\to\infty}\int \rho(\cdot,t)\,d\mu_t^{\lambda_i} = \lim_{t'\nearrow t_0}\int \rho_{X_0}\,d\mu_{t'} \qquad t<0\,,$$

where the last equality follows by the monotonicity formula. Using the monotonicity for $(\nu_t)_{t\in\mathbb{R}}$ cantered at $(0,0)$, yields that for a.e. $t<0$, ν_t is an n-dimensional integral varifold with $\mathbf{H}\in\mathscr{L}^2_{\mathrm{loc}}(\mu_t)$, and

$$\mathbf{H} + \frac{x^\perp}{-2t} = 0 \qquad \nu_t\text{-a.e. } x\,. \tag{5.7}$$

Next we show self-similarity. Define $\tilde\nu_t(A) := (-t)^{-n/2}\nu_t((-t)^{-1/2}A), t<0$. It suffices to show that $\tilde\nu_t$ is constant in t. Let $\phi\in C^2_c(\mathbb{R}^{n+1};\mathbb{R}^+)$ and $\tilde\phi(x,t) = (-t)^{n/2}\phi((-t)^{1/2}x)$. Note that

$$\frac{\partial\tilde\phi}{\partial t} = -\frac{n}{2t}\tilde\phi - \frac{1}{2t}\langle\nabla\tilde\phi, x\rangle\,.$$

By the definition of Brakke flow, we have for $t_1\le t_2<0$

$$\begin{aligned}
\int \phi\,d\tilde\nu_{t_2} - \int\phi\,d\tilde\nu_{t_1} &= \int\tilde\phi\,d\nu_{t_2} - \int\tilde\phi\,d\nu_{t_1}\\
&\le \int_{t_1}^{t_2}\int -\frac{n}{2t}\tilde\phi - \tilde\phi|\mathbf{H}|^2 + \langle\nabla\tilde\phi,\mathbf{H}\rangle - \frac{1}{2t}\langle\nabla\tilde\phi,x\rangle\,d\nu_t\,dt\\
&= \int_{t_1}^{t_2}\int -\frac{n}{2t}\tilde\phi - \frac{\tilde\phi}{2t}\langle\mathbf{H},x\rangle + \left\langle\nabla\tilde\phi,\frac{x^\perp}{2t}\right\rangle - \frac{1}{2t}\langle\nabla\tilde\phi,x\rangle\,d\nu_t\,dt\\
&= \int_{t_1}^{t_2}\int -\frac{n}{2t}\tilde\phi - \frac{\tilde\phi}{2t}\langle\mathbf{H},x\rangle - \left\langle\nabla\tilde\phi,\frac{x^T}{2t}\right\rangle\,d\nu_t\,dt\,,
\end{aligned}$$

where we used (5.7). From the first variation formula we have

$$\int -\frac{\tilde\phi}{2t}\langle\mathbf{H},x\rangle\,d\nu_t = \int\frac{1}{2t}\mathrm{div}_{T\nu_t}(\tilde\phi\,x)\,d\nu_t = \int\frac{n}{2t}\tilde\phi + \left\langle\nabla\tilde\phi,\frac{x^T}{2t}\right\rangle\,d\nu_t\,.$$

Combining with the above we see that $\int\phi\,d\tilde\nu_t$ is non-increasing in t.

Next assume without loss of generality that $\phi<\rho$ and apply the same calculation to $\rho-\phi$ (using the exponential decay of ρ and the bounded area ratios to validate the insertion of this function) to see that $\int\rho-\phi\,d\tilde\nu_t$ is also non-increasing in t.

It follows by (5.6) that $\int\phi\,d\tilde\nu_t$ is constant in t, which implies (5.4). Thus by (5.7) it follows that (5.5) holds. □

6 Level set flow and Brakke flow

6.1 The avoidance principle for Brakke flows

Theorem 6.1. *Suppose M is the space-time support of an m-dimensional integral Brakke flow $(\mu_t)_{t\in I}$ in $U \subset \mathbb{R}^N$. Let $u : U \times I \to \mathbb{R}$ be a smooth function, so that at (x_0, t_0),*

$$\frac{\partial u}{\partial t} < tr_m \nabla^2 u \,,$$

where $\nabla^2 u$ is the spacial ambient Hessian, and tr_m is the sum of the smallest m eigenvalues. Then

$$u\big|_{M\cap\{t\le t_0\}}$$

cannot have a local maximum at (x_0, t_0).

Proof. Assume otherwise. We may assume that $M = M \cap \{t \le t_0\}$ and that $u|_M$ has a strict local maximum at (x_0, t_0). (Otherwise we could replace u by $u - (d(x, x_0))^4 - |t_0 - t|^2$).

Let $P(r) = B_r(x_0) \times (t_0 - r^2, t_0]$. Choose $r > 0$ small enough so that $-r^2$ is past the initial time of the flow, $u|_{\mathcal{M}\cap\overline{P}(r)}$ has a maximum at (x_0, t_0) and nowhere else and $\frac{\partial u}{\partial t} < \mathrm{tr}_m \nabla^2 u$ on $\overline{P}(r)$. By adding a constant we can furthermore assume that $u_{\mathcal{M}\cap(\bar{P}\setminus P)} < 0 < u(x_0, t_0)$. We let $u^+ := \max\{u, 0\}$ and plug $(u^+)^4$ into the definition of Brakke flow. Thus

$$
\begin{aligned}
0 &\le \int_{B_r} (u^+)^4 \, d\mu_{t_0} = \int_{B_r} (u^+)^4 \, d\mu_{t_0} - \int_{B_r} (u^+)^4 \, d\mu_{t_0 - r^2} \\
&\le \int_{t_0 - r^2}^{t_0} \int \left(\frac{\partial}{\partial t}(u^+)^4 + \langle \mathbf{H}, \nabla (u^+)^4 \rangle - |\mathbf{H}|^2 (u^+)^4 \right) d\mu_t \, dt \\
&\le \int_{t_0 - r^2}^{t_0} \int \left(\frac{\partial}{\partial t}(u^+)^4 - \mathrm{div}_{\mathcal{M}(t)}\big(\nabla (u^+)^4\big) \right) d\mu_t \, dt \\
&= \int_{t_0 - r^2}^{t_0} \int 4\left((u^+)^3 \frac{\partial}{\partial t} u^+ - 3(u^+)^2 |\nabla^{\mathcal{M}(t)} u^+|^2 - (u^+)^3 \mathrm{div}_{\mathcal{M}(t)}\big(\nabla (u^+)\big) \right) d\mu_t \, dt \\
&\le \int_{t_0 - r^2}^{t_0} \int 4(u^+)^3 \left(\frac{\partial}{\partial t} u^+ - \mathrm{tr}_m \nabla^2 u^+ \right) d\mu_t \, dt < 0 \,,
\end{aligned}
$$

which is a contradiction. □

As a consequence of this, we obtain

Theorem 6.2 (Weak barrier principle)**.** *Let M be the space-time support of an m-dimensional integral Brakke flow in $U \subset \mathbb{R}^N$. Suppose that $t \mapsto N(t)$ is a 1-parameter family of domains in U so that $t \mapsto \partial N(t)$ is a smooth 1-parameter family of hypersurfaces. Assume that $M(t) := \{x : (x,t) \in M\} \subset N(t)$.*

If $p \in M(\tau) \cap \partial N(\tau)$, then $v(p,\tau) \ge h_m(p,\tau)$, where $v(p,\tau)$ is the speed of $\partial N(\tau)$ at p in the inward direction ν and h_m is the sum of the m smallest principal curvature of ∂N.

Proof. Let $f : U \to \mathbb{R}$ be defined by

$$
f(x,t) = \begin{cases} -\mathrm{dist}(x, \partial N(t)) : x \in N(t) \\ \mathrm{dist}(x, \partial N(t)) : x \notin N(t) \end{cases}
$$

and let $e_1, \ldots, e_{N-1}$ denote the principal curvature directions of $\partial N(t)$ at p. Then

$e_1, \dots, e_{N-1}, \nu$ is an orthonomal basis at p. We compute

$$D^2 f(p) = \begin{pmatrix} \kappa_1 & & & \\ & \ddots & & \\ & & \kappa_{N-1} & \\ & & & 0 \end{pmatrix}.$$

Set $u = e^{\alpha f}$. Then $Du = \alpha e^{\alpha f} Df$, so

$$D^2 u = \alpha^2 e^{\alpha f} Df^T Df + \alpha e^{\alpha f} D^2 f .$$

From this, we readily see that the eigenvalues of $D^2 u$ at p (note that $u(p) = 1$) are $\alpha\kappa_1, \dots \alpha\kappa_{N-1}, \alpha^2$. For α sufficiently large, we see that

$$\mathrm{tr}_m D^2 u|_p = \alpha h_m .$$

On the other hand,

$$\frac{\partial u}{\partial t} = \alpha e^{\alpha f} \frac{\partial f}{\partial t} = \alpha v(p,t) .$$

By assumption $f|_M$ has a maximum at (p, t), so the conclusion follows from the maximum principle proven above. □

Theorem 6.3 (Barrier principle for hypersurfaces)**.** *Let M be the space-time support of an n-dimensional integral Brakke flow in $\mathbb{R}^{n+1}$ and let $M(t) := \{x : (x,t) \in M\}$ denote the t-time slice of M. Suppose that $t \mapsto N(t)$ is a 1-parameter family of closed domains so that $t \mapsto \partial N(t)$ is a smooth 1-parameter family of hypersurfaces. Assume that $\partial N(t)$ is compact and connected and $v_{\partial N,in} \leq H_{\partial N,in}$ everywhere. Suppose that $M(0) \subset N(0)$ and that $\partial N(0) \setminus M(0)$ is nonempty. Then, $M(t)$ is contained in the interior of $N(t)$ for $t > 0$.*

First we prove

Lemma 6.4. *Assumptions as in the barrier principle. If $M(0) \subset N(0)$, then $M(t) \subset N(t)$.*

Proof. Let $\tilde{N}_\varepsilon(t)$ be the region with $\partial N_\varepsilon(0) = \partial N(0)$ and which flows with speed $H - \varepsilon$.

If ε is sufficiently small, this flow will be smooth on an interval comparable to that of the definition of $N(t)$. We can apply the weak maximum principle and then let $\varepsilon \to 0$. □

Proof of Theorem 6.3. Since we assume that $\partial N(0)$ has some points which are disjoint from $M(0)$, we may push $\partial N(0)$ slightly in at these points, to find a new set $\hat{N}(0)$ with $\hat{N}(0)$ smooth, and $M(0) \subset \hat{N}(0) \subsetneq N(0)$. Flow $\partial\hat{N}(0)$ by mean curvature flow; it will remain smooth for at least a short time. The classical maximum principle shows that $\partial N(t)$ and $\partial\hat{N}(t)$ immediately become disjoint. Applying the above lemma to $\hat{N}(t)$ yields the desired result. □

Remark 6.5: This only works in codimension one. Nevertheless one can see from the weak barrier principle that in higher codimension spheres with Radius $R(t) = \sqrt{R_0^2 - 2mt}$ act as barriers from the inside and from the outside.

We will now extend this to an avoidance principle for Brakke flows. We first need the following auxiliary result of Ilmanen.

Lemma 6.6 ($C^{1,1}$-interposition Lemma, [17, Lemma 4E])**.** *Given disjoint closed sets $X, Y \subset \mathbb{R}^n$, X compact, there exists a compact $C^{1,1}$ hypersurface Q and a bounded open set U such that*

(i) $X \subset U$, $Q = \partial U$, $Y \subseteq \mathbb{R}^n \setminus \overline{U}$,

(ii) $\operatorname{dist}(X, Q) + \operatorname{dist}(Q, Y) = \operatorname{dist}(X, Y)$.

This immediately implies

Theorem 6.7 (Avoidance for codimension one Brakke flows)**.** *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two n-dimensional integral Brakke flows on $\mathbb{R}^{n+1}$ defined for $t \geq 0$ and such that $\operatorname{spt}(\mathcal{M}_1(0))$ is compact and $\operatorname{spt}(\mathcal{M}_1(0)) \cap \operatorname{spt}(\mathcal{M}_2(0)) = \emptyset$. Then*

$$[0, \infty) \ni t \mapsto \operatorname{dist}(\operatorname{spt}(\mathcal{M}_1(t)) \cap \operatorname{spt}(\mathcal{M}_2(t)))$$

is strictly increasing.

Proof. Using big spheres as barriers we see that $\operatorname{spt}(\mathcal{M}_1(t))$ is compact for all $t > 0$.

By Lemma 6.6 we can choose a closed domain N such that ∂N is a $C^{1,1}$-hypersurface such that

(i) $\mathrm{spt}(\mathcal{M}_1(0)) \subset \mathrm{int}(N)$, $\mathrm{spt}(\mathcal{M}_2(0)) \subseteq \mathbb{R}^{n+1} \setminus N$,

(ii) $\mathrm{dist}(\mathrm{spt}(\mathcal{M}_1(0)), \partial N) + \mathrm{dist}(\partial N, \mathrm{spt}(\mathcal{M}_2(0))) = (\mathrm{spt}(\mathcal{M}_1(0)), \mathrm{spt}(\mathcal{M}_2(0)))$.

Let $N(t)$ be the region with $\partial N(0) = \partial N$, flowing by mean curvature. Since ∂N is $C^{1,1}$ this will exist for a short time. We can then use Theorem 6.3 to see that

$$\begin{aligned}
\mathrm{dist}(\mathrm{spt}(\mathcal{M}_1(0)) \cap \mathrm{spt}(\mathcal{M}_2(0))) &= \mathrm{dist}(\mathrm{spt}(\mathcal{M}_1(0)), \partial N) + \mathrm{dist}(\partial N, \mathrm{spt}(\mathcal{M}_2(0))) \\
&< \mathrm{dist}(\mathrm{spt}(\mathcal{M}_1(t)), \partial N(t)) + \mathrm{dist}(\partial N(t), \mathrm{spt}(\mathcal{M}_2(t))) \\
&\leq \mathrm{dist}(\mathrm{spt}(\mathcal{M}_1(t)) \cap \mathrm{spt}(\mathcal{M}_2(t)))
\end{aligned}$$

for $t > 0$. □

6.2 Level set flow

We will give a brief introduction to level set flow and discuss some connections with Brakke flow, compare [17, 18].

Definition 6.8. *A family $\{\Delta_t\}_{t\geq 0}$ of closed sets in $\mathbb{R}^{n+1}$ is a* set-theoretic subsolution *of mean curvature flow, provided that for any family $\{M_t\}_{t\in[t_0,t_1]}$ of smooth, closed hypersurfaces moving by mean curavture,*

$$\Delta_{t_0} \cap M_{t_0} = \emptyset \qquad \textit{implies} \qquad \Delta_t \cap M_t = \emptyset \quad \textit{for all } t \in [t_0, t_1]$$

or equivalently,

$$\mathit{dist}(\Delta_t, M_t) \geq \mathit{dist}(\Delta_{t_0}, M_{t_0}) \quad \textit{for } t \in [t_0, t_1]\,.$$

Note that the equivalence follows from the translation invariance of mean curvature flow.

By using the $C^{1,1}$-interposition Lemma as in the proof of the avoidance principle for Brakke flows we obtain

Lemma 6.9 (Avoidance Lemma for set-theoretic subsolutions). *Let* $\{\Delta_t\}_{t\geq 0}$, $\{\Gamma_t\}_{t\geq 0}$ *be set-theoretic subsolutions of mean curvature flow in* $\mathbb{R}^{n+1}$. *Assume*

$$\Delta_0 \cup \Gamma_0 = \emptyset, \quad \Gamma_0 \textit{ compact.}$$

Then $t \mapsto \mathit{dist}(\Delta_t, \Gamma_t)$ *is non-decreasing.*

Note that the union of set-theoretic subsolutions is trivially again a subsolution. Thus we can define

Definition 6.10 (Level-set flow). *Let* $\Delta \subset \mathbb{R}^{n+1}$ *be closed. The level-set flow* $\{F_t(\Delta)\}_{t\geq 0}$ *of* Δ *is the* maximal set-theoretic subsolution *starting such that* $\Delta_0 := F_0(\Delta) = \Delta$.

Proposition 6.11 (Basic properties). *The level-set flow is well-defined and unique, and has the following basic properties*

- *semigroup property:* $F_0(\Delta) = \Delta, F_{t+t'}(\Delta) = F_t(F_{t'}(\Delta))$,
- *commutes with translations:* $F_t(\Delta + x) = F_t(\Delta) + x$,
- *containment: if* $\Delta \subseteq \Delta'$, *then* $F_t(\Delta) \subseteq F_t(\Delta')$.

Proof. Observe first that by translation-invariance of smooth solutions, a family of closed sets $\{\Delta_t\}$ is a subsolution if and only if

$$d(\Delta_t, M_t) \geq d(\Delta_{t_0}, M_{t_0}) \quad \forall\, t \in [t_0, t_1]\,,$$

whenever $\{M_t\}_{t\in[t_0,t_1]}$ is a smooth closed mean curvature flow. Now, considering the closure of the union of all subsolutions, namely

$$F_{t'}(\Delta) = \overline{\bigcup \{\Delta_{t'} \mid \{\Delta_t\}_{t\geq 0} \text{ is a subsolution}\}}\,,$$

we see that the level-set flow exists and is unique. The basic properties follow from existence and uniqueness. □

Relation to level-set-flow as defined by Evans-Spruck and Chen-Giga-Goto, see [17]. In [11, 3], the following equation appears together with a (viscosity) definition of its weak solutions

$$(\star) \qquad \begin{cases} \partial_t u = (\delta_{ij} - \nu_i \nu_j)\nabla^2_{ij} u & \text{on } \mathbb{R}^{n+1} \times [0,\infty) \\ u(\cdot, 0) = f(\cdot) & \text{on } \mathbb{R}^{n+1} \times \{0\}, \end{cases}$$

where $\nu = Du/|Du|$. When u is smooth and $Du \neq 0$, equation($\star$) says that the level-sets of u are simultaneously moving by mean curvature. If f is continuous and all but at most one of the level-sets of f are compact, then there exists a unique u weakly solving ($\star$) (see [16, §7]). The family of level-sets $\Gamma^a_t := \{x : u(x,t) = a\}, t \geq 0$, is unique and is called the *level-set flow* (by mean curvature) of $f^{-1}(a)$. It follows trivially from the definition of weak solutions of ($\star$) (which involves tangency of u with smooth test functions, see [11]) that a level-set flow in $\mathbb{R}^{n+1}$ is a set-theoretic subsolution of mean curvature flow.

In fact, any set-theoretic subsolution that is contained within a level-set flow at $t = 0$ must remain contained within it, because otherwise it would run into some of the other level-sets if u, violating the avoidance of set-theoretic subsolutions. (Note that always one of the two sets involved in the collision is compact by the hypothesis on f). This shows the equivalence of the two definitions.

Relation to Brakke flow. From the barrier principle for hypersurfaces, Theorem 6.3 we have

Lemma 6.12. *Let M be the space-time support of an n-dimensional integral Brakke flow on $\mathbb{R}^{n+1}$. Then the family of sets $M(t) = \{x : (x,t) \in M\}$ are a set-theoretic subsolution of mean curvature flow.*

Corollary 6.13. *Let $\mathcal{M}$ be n-dimensional integral Brakke flow on $\mathbb{R}^{n+1}$ for $t \in [0,\infty)$ and let $\{\Gamma_t\}_{t\geq 0}$ be a level set flow. Then*

$$\operatorname{spt} \mu_0 \subset \Gamma_0 \quad \textit{implies} \quad \operatorname{spt} \mu_t \subset \Gamma_t$$

for all $t \geq 0$.

This implies that one can use the level-set flow to characterise possible non-uniqueness of possible Brakke flows starting at M_0. The notion used for this is called *non-fattening*.

Definition 6.14. *A* $\{\Gamma_t\}_{t\geq 0}$ *level-set flow is called* non-fattening*, provided*

$$\mathcal{H}^{n+1}\Bigg(\bigcup_{t\geq 0}\Gamma_t \times \{t\}\Bigg) = 0\,.$$

Note that this implies that for the level-set flow $\{\Gamma_t^a\}_{t\geq 0}$ as defined by Evans-Spruck or Chen-Giga-Goto at most countably levels $\{u(x,t) = a\}$ could be fattening. This shows that non-fattening is a generic condition. We have the following equivalence:

Lemma 6.15 ([18, §11.4], [12, 4.2])**.** *If* $\mathcal{H}^n(\Gamma_0) < \infty$*, then the level set flow* $\{\Gamma_t\}_{t\geq 0}$ *is non-fattening if and only if* $\mathcal{H}^n(\Gamma_t) < \infty$ *for all* $t \geq 0$*.*

Remark 6.16: It follows by work of Hershkovits-White [14] and the resolution of the mean-convex neighborhood conjecture by Choi-Haslhofer-Hershkovits [5] and Choi-Haslhofer-Hershkovits-White [6] that if a unit-regular, cyclic mod 2, n-dimensional integral Brakke flow $\mathcal{M}$ in $\mathbb{R}^{n+1}$, starting at a compact, smooth, embedded hypersurface M_0 has only multiplicity one spherical and neck-pinch (i.e. of type $\mathbb{S}^{n-1}\times\mathbb{R}$) singularities, then the level-set flow of M_0 is non-fattening. It then follows from [4, Corollary F.5] that the unit-regular, cyclic mod 2, n-dimensional integral Brakke flow starting at M_0 is unique.

Bibliography

[1] William K. Allard, *On the first variation of a varifold*, Ann. of Math. (2) **95** (1972), 417–491.

[2] Kenneth Brakke, *The motion of a surface by its mean curvature*, Princeton Univ. Press, 1978.

[3] Yun Gang Chen, Yoshikazu Giga, and Shun'ichi Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Differential Geom. **33** (1991), no. 3, 749–786.

[4] Otis Chodosh, Kyeongsu Choi, Christos Mantoulidis, and Felix Schulze, *Mean curvature flow with generic initial data*, https://arxiv.org/abs/2003.14344 (2020).

[5] Kyeongsu Choi, Robert Haslhofer, and Or Hershkovits, *Ancient low entropy flows, mean convex neighborhoods, and uniqueness*, 2018, arXiv:1810.08467.

[6] Kyeongsu Choi, Robert Haslhofer, Or Hershkovits, and Brian White, *Ancient asymptotically cylindrical flows and applications*, https://arxiv.org/abs/1910.00639 (2019).

[7] Julie Clutterbuck, Oliver C. Schnürer, and Felix Schulze, *Stability of translating solutions to mean curvature flow*, Calc. Var. Partial Differential Equations **29** (2007), no. 3, 281–293.

[8] Tobias H. Colding and William P. Minicozzi, II, *Generic mean curvature flow I: generic singularities*, Ann. of Math. (2) **175** (2012), no. 2, 755–833.

[9] Manfredo Perdigão do Carmo, *Riemannian geometry*, Mathematics: Theory &

Applications, Birkhäuser Boston, Inc., Boston, MA, 1992, Translated from the second Portuguese edition by Francis Flaherty.

[10] Lawrence C. Evans, *Partial differential equations*, second ed., Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 2010.

[11] Lawrence C. Evans and Joel Spruck, *Motion of level-sets by mean curvature I*, J. Diff. Geom. **33** (1991), 635–681.

[12] L.C. Evans and J. Spruck, *Motion of level-sets by mean curvature III*, J. Geom. Anal. **2** (1992), no. 2, 121–150.

[13] Richard S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Diff. Geom. **17** (1982), 255–306.

[14] Or Hershkovits and Brian White, *Non-fattening of mean curvature flow at singularities of mean convex type*, 2014, `arXiv:1704.00431`.

[15] Gerhard Huisken and Alexander Polden, *Geometric evolution equations for hypersurfaces*, Calculus of variations and geometric evolution problems (Cetraro, 1996), Lecture Notes in Math., vol. 1713, Springer, Berlin, 1999, pp. 45–84.

[16] Tom Ilmanen, *Generalized flow of sets by mean curvature on a manifold*, Indiana Univ. Math. J. **41** (1992), no. 3, 671–705.

[17] ______, *The level-set flow on a manifold*, Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990), Proc. Sympos. Pure Math., vol. 54, Amer. Math. Soc., Providence, RI, 1993, pp. 193–204.

[18] ______, *Elliptic regularization and partial regularity for motion by mean curvature*, Mem. Amer. Math. Soc. **108** (1994), no. 520, x+90.

[19] Leon Simon, *Lectures on geometric measure theory*, Centre for Mathematical Analysis, Australian National Unversity, 1983.

[20] Xu-Jia Wang, *Convex solutions to the mean curvature flow*, Ann. of Math. (2) **173** (2011), no. 3, 1185–1239.

[21] Brian White, *Mean curvature flow with boundary*, https://arxiv.org/abs/1901.03008 (2019).