Uniqueness of Weak Solutions to the Ricci Flow and Topological Applications

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(based on joint work with Bruce Kleiner, NYU)

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Structure of Course

1. Introduction: Preliminaries + Background on Ricci flow, Blow-up analysis of singularities, Precise statement of the main results
2. Analytical aspects I: Local stability analysis
3. Analytical aspects II: Comparing singular Ricci flows, Proof of the uniqueness and stability result
4. Topological aspects: Continuous families of singular Ricci flows, Proof of the topological applications

Main results, Overview

Analytic result:

- Given any closed (M^3, g), there is a unique/canonical, weak Ricci flow "through singularities" $\mathcal{M}^{(M,g)}$ starting from (M, g).
- This flow depends continuously on g.

Topological applications:

- For any closed M^3, the space $\mathrm{Met}_{PSC}(M)$ of Riemannian metrics with positive scalar curvature is either empty or contractible.
- $\mathrm{Diff}(S^3/\Gamma) \simeq \mathrm{Isom}(S^3/\Gamma)$
- $\mathrm{Diff}(M) \simeq \mathrm{Isom}(M)$ if M is hyperbolic
- similar results if $M = S^2 \times S^1$, infranil, etc.

(based on joint work with Bruce Kleiner)

Lecture I

A Statement of the topological results

B Ricci flow, Motivation, Singularity analysis, RF with surgery

C Singular Ricci flows, Statement of the main uniqueness result

Lecture I.A

Statement of the topological results

Basic definitions

M (mostly) 3-dimensional, compact, orientable manifold

Recall: The topology of 3-manifolds is sufficiently well understood due to the resolution of the Poincaré and Geometrization Conjectures by Perelman, using Ricci flow.

Main objects of study:

- $\mathrm{Met}(M)$: space of Riemannian metrics on M
- $\mathrm{Met}_{PSC}(M) \subset \mathrm{Met}(M)$: subset of metrics with positive scalar curvature
- $\mathrm{Diff}(M)$: space of diffeomorphisms $\phi : M \to M$

…each equipped with the C^∞-topology.

Goal: Classify these spaces up to homotopy (using Ricci flow)!

$\mathrm{Met}(M)$ is contractible

Space of PSC-metrics

Main Result 1:

Ba., Kleiner 2019

$\mathrm{Met}_{PSC}(M)$ is either contractible or empty.

History:

- true in dimension 2 (via Uniformization Theorem or Ricci flow (see later))
- Hitchin 1974; Gromov, Lawson 1984; Botvinnik, Hanke, Schick, Walsh 2010: Further examples with $\pi_i(\mathrm{Met}_{PSC}(M^n)) \neq 1$ for certain (large) i, n.
- Marques 2011 (using Ricci flow with surgery):
  $\mathrm{Met}_{PSC}(M^3)/\operatorname{Diff}(M^3)$ is path-connected,
  $\mathrm{Met}_{PSC}(S^3)$ is path-connected

Diffeomorphism groups

Smale 1958: $O(3) \simeq \mathrm{Diff}(S^2)$

Smale Conjecture: $O(4) \simeq \mathrm{Diff}(S^3)$
proven by Hatcher in 1983

For a general spherical space form $M = S^3/\Gamma$ consider the injection

$$\mathrm{Isom}(M) \longrightarrow \mathrm{Diff}(M)$$

Generalized Smale Conjecture

This map is a homotopy equivalence.

- Verified for a handful of other spherical space forms, but open e.g. for $\mathbb{R}P^3$.
- All proofs so far are purely topological and technical. No uniform treatment.

Main Result 2:

Theorem (Ba., Kleiner 2019)

The Generalized Smale Conjecture is true.

Remarks:

- Proof via Ricci flow (first purely topological application of Ricci flow since Perelman's work $\sim$ 15 years ago).
- Uniform treatment of all cases.
- Alternative proof in the S^3-case (Smale Conjecture).
- There are two proofs:
    - "Short" proof (Ba., Kleiner 2017): GSC if $M \not\approx S^3, \mathbb{R}P^3$, M hyperbolic, assuming the Smale Conjecture for S^3
    - Long proof (Ba., Kleiner 2019): full GSC and $S^2 \times \mathbb{R}$-cases

Similar techniques imply results in non-spherical case:

- If M is closed and hyperbolic, then $\mathrm{Isom}(M) \simeq \mathrm{Diff}(M)$.
  (topological proof by Gabai 2001)
- If (M, g) is aspherical and geometric and g has maximal symmetry, then $\mathrm{Isom}(M) \simeq \mathrm{Diff}(M)$.
  (new in non-Haken infranil case)
- $\mathrm{Diff}(S^2 \times S^1) \simeq O(2) \times O(3) \times \Omega O(3)$
  (topological proof by Hatcher)
- $\mathrm{Diff}(\mathbb{R}P^3 \# \mathbb{R}P^3) \simeq O(1) \times O(3)$
  (topological proof by Hatcher)

Connection between $\mathrm{Diff}(M) \longleftrightarrow \mathrm{Met}(M)$

Lemma

For any $g \in \mathrm{Met}_{K\equiv\pm1}(M)$:

$$\mathrm{Isom}(M,g) \simeq \mathrm{Diff}(M) \qquad \Longleftrightarrow \qquad \mathrm{Met}_{K\equiv\pm1}(M) \;\text{ contractible}$$

Proof: Fiber bundle

$$\begin{aligned} \mathrm{Isom}(M,g) \longrightarrow \mathrm{Diff}(M) &\longrightarrow \mathrm{Met}_{K\equiv\pm1}(M) \\ \phi &\longmapsto \phi^* g \end{aligned}$$

Apply long exact homotopy sequence:

$$\begin{aligned} 0 = \pi_{i+1}(\mathrm{Met}_{K\equiv\pm1}(M)) & \\ \longrightarrow \pi_i(\mathrm{Isom}(M,g)) \longrightarrow \pi_i(\mathrm{Diff}(M)) &\longrightarrow \pi_i(\mathrm{Met}_{K\equiv\pm1}(M)) = 0 \end{aligned}$$

This reduces both results to:

Theorem (Ba., Kleiner 2019)

$\mathrm{Met}_{PSC}(M)$ and $\mathrm{Met}_{K\equiv\pm1}(M)$ are each either contractible or empty.

Lecture I.B

Ricci flow, Motivation, Singularity analysis, RF with surgery

Ricci flow equation $(M, g(t)), \quad t \in [0, T)$

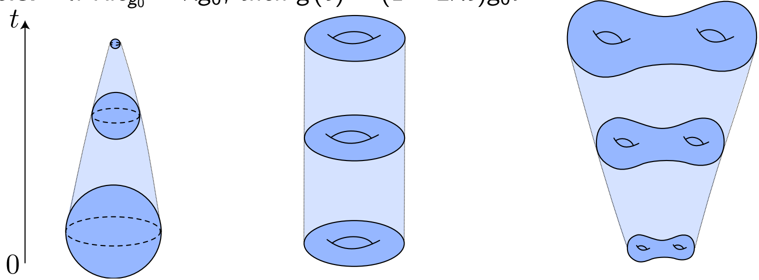
$$\partial_t g(t) = -2\,\mathrm{Ric}_{g(t)}, \qquad g(0) = g_0 \tag{*}$$

(in (harmonic) coordinates: “ $\partial_t g_{ij} = \triangle g_{ij} + \dots$ ”*)*

Hamilton 1982

- $(*)$ has a unique solution $(g(t))_{t\in[0,T)}$ for maximal $T \in (0, \infty]$.
- If $T < \infty$, then “$g(t)$ develops a singularity at time T”, i.e. the curvature $|\mathrm{Rm}|$ blows up as $t \nearrow T$.

Example: If $\mathrm{Ric}_{g_0} = \lambda g_0$, then $g(t) = (1 - 2\lambda t) g_0$.



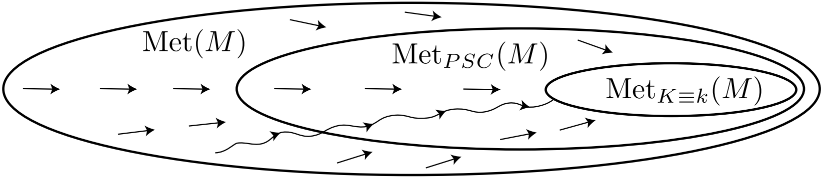
$\lambda > 0$ $\qquad$ $\lambda = 0$ $\qquad$ $\lambda < 0$

Ricci flow in 2D

Hamilton, Chow: On $M = S^2$ for any initial condition g_0 we have

$$T = \frac{\mathrm{vol}(S^2, g_0)}{8\pi}, \qquad (T-t)^{-1} g(t) \longrightarrow g_{\mathrm{round}}$$

Interpretation on the space of metrics:



- Preservation of positive scalar curvature (in all dimensions)
- $\rightsquigarrow$ deformation retractions from $\mathrm{Met}(S^2)$ and $\mathrm{Met}_{PSC}(S^2)$ onto $\mathrm{Met}_{K\equiv 1}(S^2)$

Theorem

$\mathrm{Met}_{PSC}(S^2) \simeq \mathrm{Met}_{K\equiv 1}(S^2) \simeq \mathrm{Met}(S^2) \simeq *$
Therefore $\mathrm{Diff}(S^2) \simeq O(3)$.

Ricci flow in 3D

Difficulties:

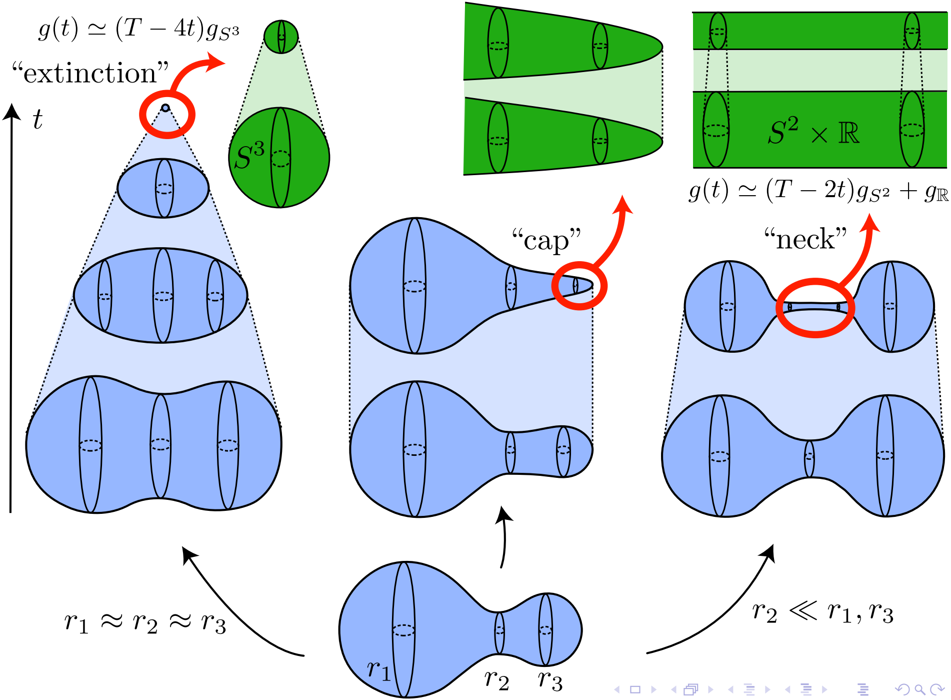
- Flow may incur non-round and non-global singularities.
- Necessary to extend the flow past the first singular time (surgeries).
- Continuous dependence on initial data?

Key analytical result:

Theorem (Ba., Kleiner, 2016)

Any (compact) 3-dimensional (M^3, g) can be evolved into a **unique** (canonical), weak Ricci flow defined for all $t \geq 0$ that "flows through singularities" and we have continuous dependence on the initial data.

Example: rotationally symmetric dumbbell


$g(t) \simeq (T-4t)g_{S^3}$
“extinction”
t
S^3
“cap”
$S^2 \times \mathbb{R}$
$g(t) \simeq (T-2t)g_{S^2} + g_{\mathbb{R}}$
“neck”
$r_1 \approx r_2 \approx r_3$
$r_2 \ll r_1, r_3$
r_1
r_2
r_3

Singularity analysis, general case

General case (no rotational symmetry):

Perelman 2002, very imprecise form

"These are (essentially) all 3D singularities"

To make this statement more precise, we have to discuss:

- the No Local Collapsing Theorem
- geometric convergence of RFs (blow-up analysis of singularities → singularity models)
- (qualitative) classification of singularity models

No Local Collapsing

Consider a general Ricci flow $(M^n, (g_t)_{t\in[0,T)})$

Goal: "Rule out $\approx S^1(\varepsilon) \times \Sigma^{n-1}$-singularity models"

No Local Collapsing Theorem (Perelman 2002)

There is a constant $\kappa(n, T, g_0) > 0$ such that $(M^n, (g_t)_{t\in[0,T)})$ is κ-noncollapsed at scales < 1, i.e.:

For any $(x,t) \in M \times [0,T)$ and $r \in (0,1)$:

$$|\mathrm{Rm}|(\cdot, t) < r^{-2} \quad \text{on} \quad B(x,t,r) \qquad \Longrightarrow \qquad |B(x,t,r)|_t \geq \kappa r^n$$

Corollary

$$|\mathrm{Rm}|(\cdot, t) < r^{-2} \quad \text{on} \quad B(x,t,r) \qquad \Longrightarrow \qquad inj(x,t) > c(\kappa) r^n$$

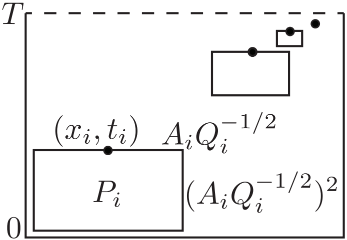
Blow-up analysis

Choose $(x_i, t_i) \in M \times [0, T)$ s.t.:

- $Q_i := |\mathrm{Rm}|(x_i, t_i) \to \infty$
- For some $A_i \to \infty$ we have:

$$|\mathrm{Rm}| \leq CQ_i \quad \text{on} \quad P_i = P(x_i, t_i, A_i Q_i^{-1/2})$$

where $P(x, t, r) = B(x, t, r) \times [t - r^2, t]$
("parabolic ball")



Parabolic rescaling: $g^i_t := Q_i g_{Q_i^{-1} t + t_i}$

- Defined for $t \in [-Q_i t_i, 0]$, where $Q_i t_i \to \infty$
- $|\mathrm{Rm}| \leq C$ on $P(x_i, 0, A_i)$
- NLC $\implies \mathrm{inj}(x_i, 0) > c > 0$

After passing to a subsequence:

$$(M, g^i_0, x_i) \xrightarrow[i \to \infty]{C^\infty - CG} (\overline{M}, \overline{g}, \overline{x})$$

i.e. $\exists$ diffeomorphisms onto their images $\psi_i : B^{\overline{M}}(\overline{x}, 0, A'_i) \to M_i$,
$\psi_i(\overline{x}) = x_i$, $A'_i \to \infty$ s.t.

$$\psi_i^* g^i_0 \xrightarrow[i \to \infty]{C^\infty_{loc}} \overline{g}$$

$$(M, g_0^i, x_i) \xrightarrow[i\to\infty]{C^\infty - CG} (\overline{M}, \overline{g}, \overline{x})$$

i.e. $\exists$ diffeomorphisms onto their images $\psi_i : B^{\overline{M}}(\overline{x}, 0, A_i') \to M_i$, $\psi_i(\overline{x}) = x_i$, $A_i' \to \infty$ s.t.

$$\psi_i^* g_0^i \xrightarrow[i\to\infty]{C^\infty_{loc}} \overline{g}$$

Using $|\mathrm{Rm}| < C$ on $P(x_i, 0, A_i)$, we can show that $|\partial^m \psi_i^* g_t^i|$ are locally uniformly bounded for all i. So after passing to a subsequence

$$\psi_i^* g_t^i \xrightarrow[i\to\infty]{C^\infty_{loc}} \overline{g}_t,$$

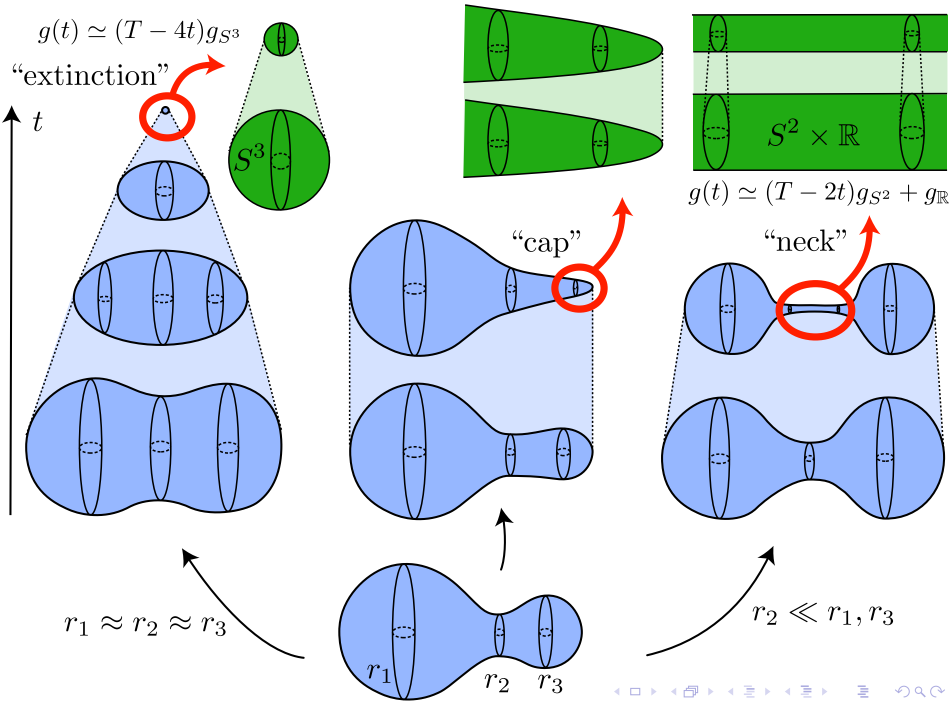
where $(\overline{g}_t)_{t\in(-\infty,0]}$ is an ancient Ricci flow with $\overline{g}_0 = \overline{g}$.

Hamilton's convergence of RFs

For a subsequence

$$(M, (g_t^i)_{t\in[-t_i Q_i, 0]}, x_i) \xrightarrow[i\to\infty]{C^\infty - HCG} (\overline{M}, (\overline{g}_t)_{t\in(-\infty,0]}, \overline{x})$$

"$(\overline{M}, (\overline{g}_t)_{t\in(-\infty,0]}, \overline{x})$ models the flow near (x_i, t_i) for large i"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r_1
r_2
r_3

General case (no rotational symmetry):

Perelman 2002, version 2

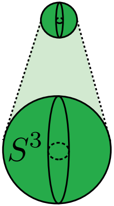
All blow-up models $(\overline{M}, (\overline{g}_t)_{t\in(-\infty,0]})$ of $(M, (g_t)_{t\in[0,T)})$ are κ-solutions.

κ-solution: ancient flow $(\overline{M}, (\overline{g}_t)_{t\in(-\infty,0]})$ s.t.

- $sec \geq 0, \quad R > 0, \quad |\mathrm{Rm}| < C \qquad$ on $\qquad \overline{M} \times (-\infty, 0]$
- κ-noncollapsed at all scales:
  $|\mathrm{Rm}| < r^{-2}$ on $B(x,t,r) \qquad \Longrightarrow \qquad |B(x,t,r)|_t \geq \kappa r^3$

Qualitative classification of κ-solutions (after Perelman)

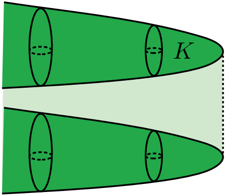
“extinction”



$\overline{M} \approx S^3/\Gamma$

round if $\Gamma \neq 1, \mathbb{Z}_2$
otherwise, possibly
ellipsoidal

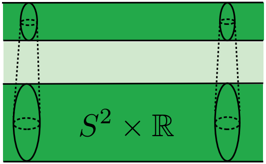
“cap”



$\overline{M} \approx \mathbb{R}^3$

$\overline{M} - K \approx S^2 \times (0, \infty)$
cylindrical near ∞

“neck”



$\overline{M} \approx S^2 \times \mathbb{R} \,/\, \Gamma$

$\overline{g}_t = -2tg_{S^2} + g_{\mathbb{R}}$
$\Gamma = 1, \mathbb{Z}_2$

Example in “cap” case: Bryant soliton $(M_{Bry}, (g_{Bry,t})_{t\in\mathbb{R}}, x_{Bry})$

$M_{Bry} = \mathbb{R}^3, x_{Bry} = 0$

$$g_{Bry,t} = dr^2 + f_t^2(r) g_{S^2}$$

$$f_t(r) \sim \sqrt{r} \qquad \text{as} \qquad r \to \infty$$

steady soliton equation: $\mathrm{Ric} = \nabla^2 f = \mathcal{L}_{\frac{1}{2}\nabla f} g \quad \Longrightarrow \quad g_{Bry,t} = \Phi_t^* g_{Bry,0}$

Brendle 2011

κ-solution + steady soliton $\Longrightarrow$ homothetic to the Bryant soliton.

Hamilton + Brendle

$\partial_t R(x,t) \geq 0$ on any κ-solution $(\overline{M},(g_t))$

Equality $\Longrightarrow$ steady soliton

$\Longrightarrow (\overline{M},(g_t),x)$ homothetic to $(M_{Bry},(g_{Bry,t}),x_{Bry})$

Brendle, Angenent, Daskalopoulos, Sesum, Kleiner, Ba. $\sim$ 2019

Every κ-solution is homothetic to a quotient of the round sphere, cylinder, the Bryant soliton or Perelman's ellipsoid.

(Not needed for Uniqueness Theorem)

Singularity analysis, reworded

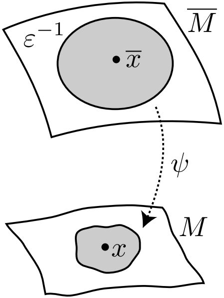
(M, g) Riemannian manifold, $x \in M$ point

- curvature scale: $\rho(x) = (\frac{1}{3} R(x))^{-1/2}$
- $(\overline{M}, \overline{g}, \overline{x})$ pointed Riem. manifold (model space)
- $(\overline{M}, \overline{g}, \overline{x})$ local ε-model at x if there is a diffeo onto its image

  $$\psi : B(\overline{x}, \varepsilon^{-1}) \longrightarrow M$$

  such that $\psi(\overline{x}) = x$ and

  $$\big\| \rho^{-2}(x) \psi^* g - \overline{g} \big\|_{C^{[\varepsilon^{-1}]}} < \varepsilon.$$



$(M, (g_t)_{t \in [0,T)})$ Ricci flow

- ... satisfies ε-canonical neighborhood assumption at scales $< r_0$ if all (x, t) with $\rho(x, t) < r_0$ are locally ε-modeled on the final time-slice $(\overline{M}, \overline{g}_0, \overline{x})$ of a pointed κ-solution.

Perelman 2002, version 3

$(M, (g_t)_{t \in [0,T)})$ satisfies the ε-canonical nbhd assumption at scales $< r(\varepsilon)$.

Ricci flow with surgery

Ricci flow with surgery:

$$(M_1, g^1_t), t \in [0, T_1],$$
$$(M_2, g^2_t), t \in [T_1, T_2],$$
$$(M_3, g^3_t), t \in [T_1, T_2], \quad \ldots$$

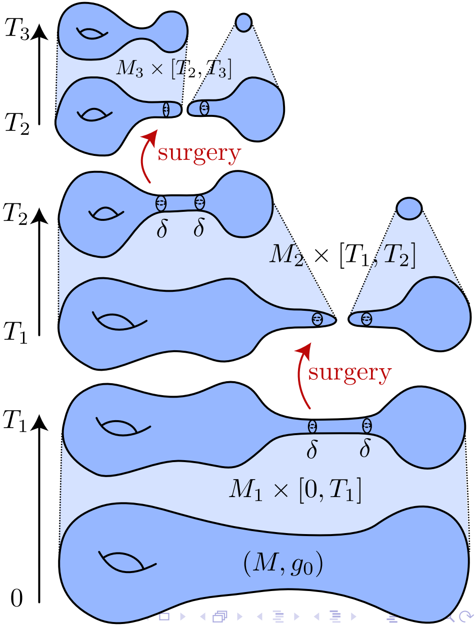
transition maps

$$U_i^- \subset M_i \qquad U_i^+ \subset M_{i+1}$$
$$\varphi_i : (U_i^-, g^i_{T_i}) \xrightarrow{\cong} (U_i^+, g^{i+1}_{T_i})$$

surgery scale $\approx \delta \ll 1$

Theorem (Perelman 2003)

This process can be continued indefinitely.
No accumulation of T_i.

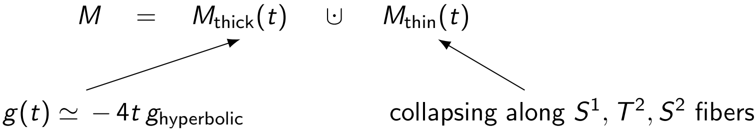


If this flow goes extinct in finite time, then $M \approx \# S^3/\Gamma_i \# k(S^2 \times S^1)$:

Poincaré Conjecture (Perelman 2003)

$$\pi_1(M) = 1 \quad \Longrightarrow \quad M \approx S^3.$$

If this flow exists for all times, then for $t \gg 1$:



$$M \quad = \quad M_{\text{thick}}(t) \quad \dot\cup \quad M_{\text{thin}}(t)$$

$g(t) \simeq -4t\, g_{\text{hyperbolic}}$

collapsing along S^1, T^2, S^2 fibers

Geometrization Conjecture (Perelman 2003)

Every closed 3-manifold can be cut along embedded, incompressible copies of S^2, T^2 into pieces, which admit a homogeneous geometry.

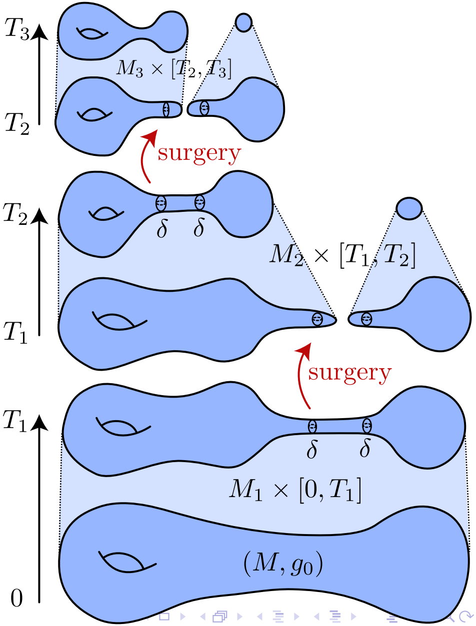
Ricci flow with surgery

Drawback:

surgery process is not canonical (depends on surgery parameters)

Perelman:

- *It is likely that [. . .] one would get a canonically defined Ricci flow through singularities, but at the moment I don't have a proof of that.*
- *Our approach [. . .] is aimed at eventually constructing a canonical Ricci flow, [. . .] - a goal, that has not been achieved yet in the present work.*



Lecture I.C

Singular Ricci flows, Statement of the main uniqueness result

Ba., Kleiner, Lott 2016

"Perelman's Conjecture is true":

- There is a reasonable definition of weak Ricci flows "through singularities".
- We have existence and uniqueness within this class.

Comparison with Mean Curvature Flow:

- Existence of weak flows "through singularities":
  Level Set Flow, Brakke Flow
- General case: fattening $\cong$ non-uniqueness
- Mean convex case: non-fattening $\cong$ uniqueness
- 2-convex case: uniqueness + weak flow is limit of MCFs with surgery as surgery scale $\delta \to 0$

Ricci flow with surgery

Ricci flow with surgery:

$$(M_1, g^1_t), t \in [0, T_1], \quad (M_2, g^2_t), t \in [T_1, T_2], \quad (M_3, g^3_t), t \in [T_1, T_2], \quad \ldots$$

transition maps

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$$\varphi_i : (U_i^-, g^i_{T_i}) \xrightarrow{\cong} (U_i^+, g^{i+1}_{T_i})$$

surgery scale $\approx \delta \ll 1$

Theorem (Perelman 2003)

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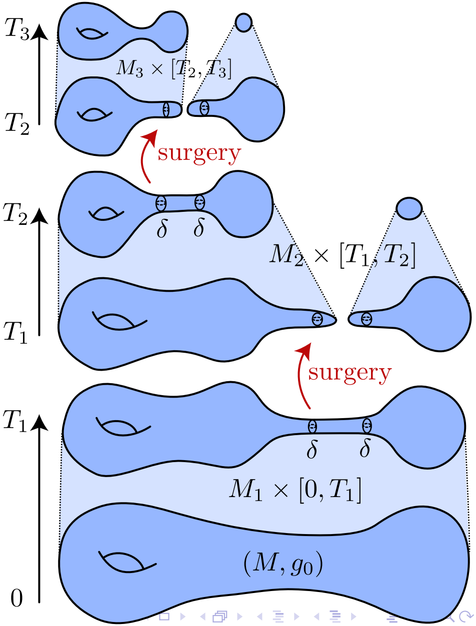


Space-time picture

- Space-time 4-manifold:

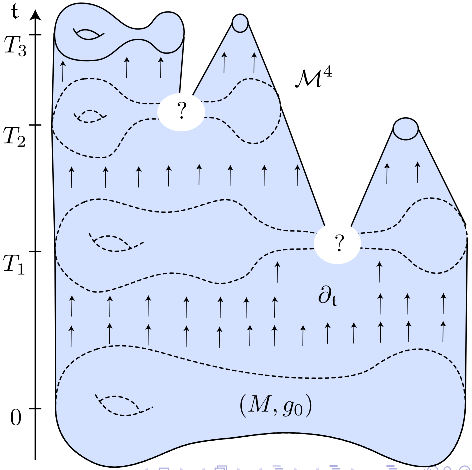
$$\mathcal{M}^4 = \big(M_1 \times [0, T_1] \;\cup_{\varphi_1}\; M_2 \times [T_1, T_2] \;\cup_{\varphi_2}\; M_3 \times [T_2, T_3] \;\cup_{\varphi_3}\; \ldots \big) - \mathcal{S}$$

$$\mathcal{S} = (M_1 \times \{T_1\} - U_1^-) \cup (M_2 \times \{T_1\} - U_1^+) \cup \ldots \quad \text{(surgery points)}$$

- Time function: $\mathfrak{t} : \mathcal{M} \to [0, \infty)$.
- Time-slice: $\mathcal{M}_t = \mathfrak{t}^{-1}(t)$
- Time vector field: $\partial_{\mathfrak{t}}$ on $\mathcal{M}$ (with $\partial_{\mathfrak{t}} \cdot \mathfrak{t} = 1$).
- Metric g: on the distribution $\{d\mathfrak{t} = 0\} \subset T\mathcal{M}$
- Ricci flow equation: $\mathcal{L}_{\partial_{\mathfrak{t}}} g = -2\,\mathrm{Ric}_g$

$(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ is called a Ricci flow spacetime.

Note: there are "holes" at scale $\approx \delta$



Space-time picture

The spacetime $(\mathcal{M}, \mathfrak{t}, \partial_\mathfrak{t}, g)$

- …satisfies the ε-canonical neighborhood assumption at scales $(C\delta, r_\varepsilon)$
- …is $C\delta$-complete,

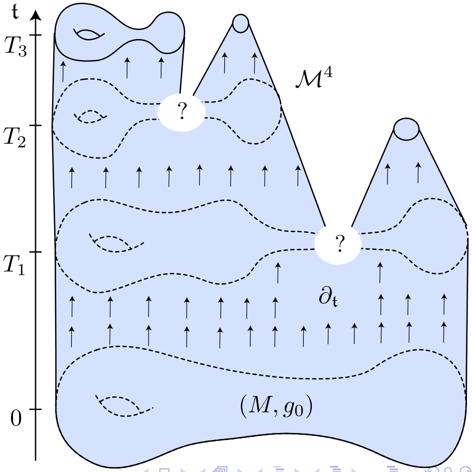
Definition

$\mathcal{M}$ is r_0-complete if the following is true: Suppose that $\gamma : [0, s_0) \to \mathcal{M}$ is a curve and:

- $\gamma([0, s_0)) \subset \mathcal{M}_{t_0}$ for some t_0 and $length_{t_0}(\gamma) < \infty$, OR
- γ is a trajectory of $\pm\partial_\mathfrak{t}$

Then if $\inf_{s\in[0,s_0)} \rho(\gamma(s)) > r_0$, then $\lim_{s\nearrow s_0} \gamma(s)$ exists.

This implies that $\{\rho \geq r', \mathfrak{t} \leq T\} \subset \mathcal{M}$ is compact if $r' > r_0$



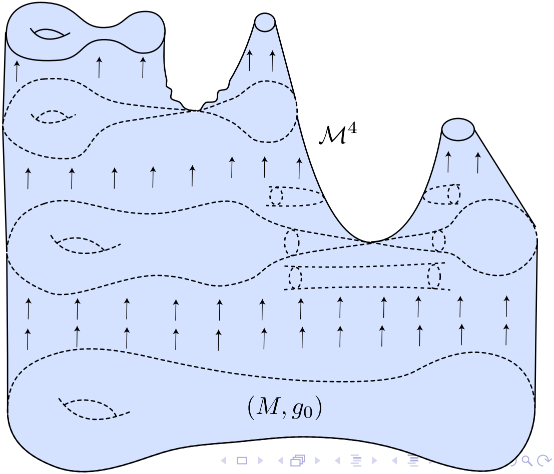
Kleiner, Lott 2014: Compactness theorem and $\delta_i \to 0 \rightsquigarrow \mathcal{M}^{\delta_i} \to \mathcal{M}$
$\Longrightarrow$ existence of (weak) singular Ricci flow starting from any (M, g_0)

Singular Ricci flow: Ricci flow spacetime $\mathcal{M}$ that:

- is 0-complete (i.e. "surgery scale $\delta = 0$")
- satisfies the ε-canonical neighbhd assumption below some scale $r(\varepsilon)$ for any ε

Remarks:

- $\mathcal{M}$ is smooth everywhere and not defined at singularities
- singular times may accummulate



Uniqueness

Theorem (Ba., Kleiner, 2017)

There is a constant $\varepsilon_{\text{can}} > 0$ such that:
Every Ricci flow space-time $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ is uniquely determined (up to isometry) by its initial time-slice $(\mathcal{M}_0, g_0)$, provided that it

- is 0-complete and
- satisfies the ε_{can}-canonical neighborhood assumption below some positive scale.

So it is enough to define singular Ricci flow via ε_{can}-canonical neighborhood assumption for some *universal* ε_{can}.

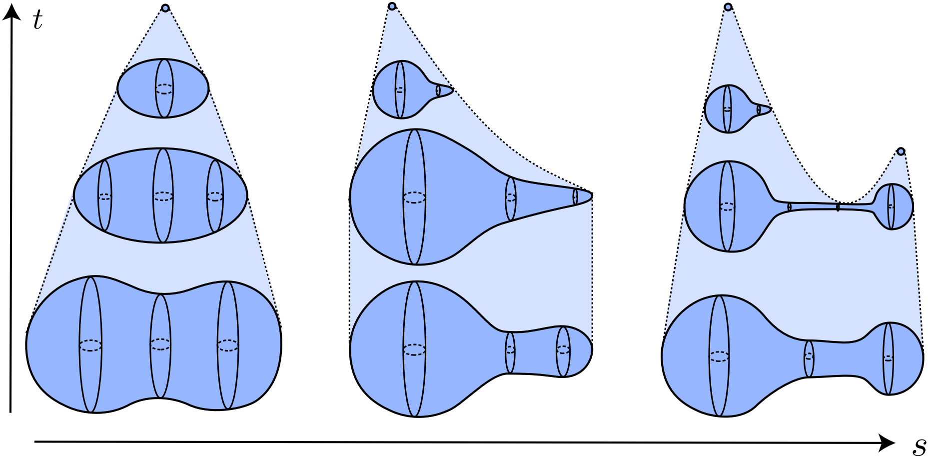
Corollary

For every compact (M^3, g_0) there is a unique, canonical singular Ricci flow space-time $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ with $\mathcal{M}_0 = (M^3, g_0)$.

Continuity of RF space-times

continuous family of metrics $(g_s)_{s\in[0,1]}$ on M

$\rightsquigarrow \quad \{\mathcal{M}_s\}_{s\in[0,1]}$ canonical RF space-times



Theorem (Ba., Kleiner)

Every continuous family $(g_s)_{s\in\Omega}$ of Riemannian metrics on a compact manifold M^3 gives rise to a "continuous family of Ricci flow space-times".

More on this later . . .

Structure of proof

Theorem (Ba., Kleiner, 2017)

There is a constant $\varepsilon_{\text{can}} > 0$ such that:
Every Ricci flow space-time $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ is uniquely determined (up to isometry) by its initial time-slice $(\mathcal{M}_0, g_0)$, provided that it

- is 0-complete and
- satisfies the ε_{can}-canonical neighborhood assumption below some positive scale.

Structure of proof

1. Blow-up analysis of almost singular part (Lectures 1+2)
2. Linear stability theory (Lecture 2)
3. Construction of comparison map + Spatial Extension Principle (Lecture 3)

Lecture II

A (Strong) ε-necks

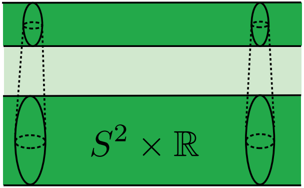
B Setup of the Toy Case

C Uniqueness in the non-singular case

D Local stability analysis

Lecture II.A

(Strong) ε-necks



Recall: $\rho = (\frac{1}{3} R)^{-1/2}$

shrinking cylinder: $g_{S^2 \times \mathbb{R}, t} = (\frac{2}{3} - 2t) g_{S^2} + g_{\mathbb{R}}$

$$\rho(\cdot, 0) \equiv 1, \quad \rho(\cdot, -1) \equiv 2$$

(M, g) Riemannian manifold

ε-neck (at scale r): $U \subset M$ such that there is a diffeomorphism $\psi : S^2 \times (-\varepsilon^{-1}, \varepsilon^{-1}) \to U$ with

$$\left\| r^{-2} \psi^* g - g_{S^2 \times \mathbb{R}, 0} \right\|_{C^{[\varepsilon^{-1}]}(S^2 \times (-\varepsilon^{-1}, \varepsilon^{-1}))} < \varepsilon$$

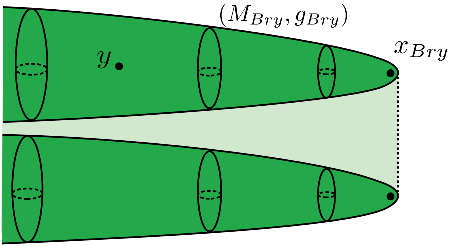
$\psi(S^2 \times \{0\})$ central 2-sphere, consisting of centers of U

$(M, (g_t))$ Ricci flow

strong ε-neck (at scale r): as before, but

$$\left\| r^{-2} \psi^* g_{r^2 t + t_0} - g_{S^2 \times \mathbb{R}, t} \right\|_{C^{[\varepsilon^{-1}]}(S^2 \times (-\varepsilon^{-1}, \varepsilon^{-1}) \times [-1, 0])} < \varepsilon$$

Example: Bryant soliton $(M_{Bry}, g_{Bry}, x_{Bry})$



If $d_t(y, x_{Bry}) > C_{Bry}(\varepsilon)$, then (y, t) is a center of a strong ε-neck.

Lemma

For a general RF $(M, (g_t)_{t\in[0,T)})$ and $(y, t) \in M \times [0, T)$ we have:
There is $\varepsilon'(\varepsilon) > 0$ such that if

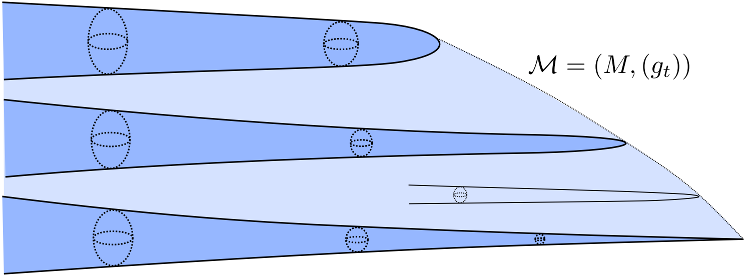
- (y, t) is a center of an ε'-neck and
- the flow satisfies the ε'-canonical neighborhood assumption below scale $\varepsilon^{-1}\rho(y, t)$,

then (y, t) is also a center of a *strong* ε-neck.

Lecture II.B

Setup of the Toy Case

We will only consider Ricci flow spacetimes $\mathcal{M}$ with the following properties:

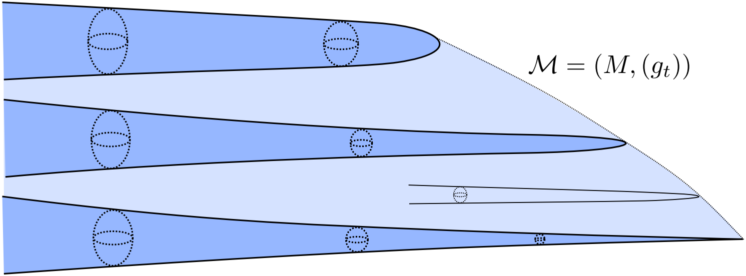


1. $\mathcal{M}_{(0,T)}$ comes from a non-singular, conventional RF $(M, (g_t)_{t \in (0,T)})$ that is κ-noncollapsed at scales < 1 and $R > 0$.
2. $\mathcal{M}_0$ can be compactified by adding a single point. $\mathcal{M}_0 \cong (M - X, g_0 := \lim_{t \searrow 0} g_t)$ for some $X \subset M$.
3. $\rho_{\min}(t) := \min_M \rho(\cdot, t) = \rho(x_t, t)$ is weakly increasing.

Note: (3) $\Longrightarrow$ $\partial_t R(x_t, t) \leq 0$ for all $t > 0$

$$\Longrightarrow \quad (M, \rho^{-2}(x_t, t) g_t, x_t) \xrightarrow[t \searrow 0]{C^\infty - HCG} (M_{Bry}, g_{Bry}, x_{Bry})$$

We will only consider Ricci flow spacetimes $\mathcal{M}$ with the following properties:

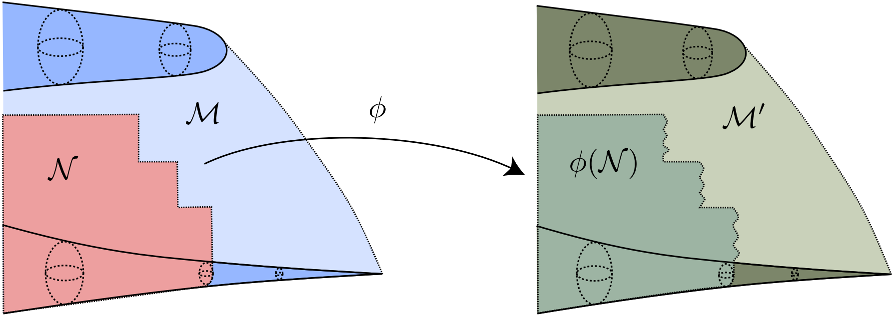


1. $\mathcal{M}_{(0,T)}$ comes from a non-singular, conventional RF $(M, (g_t)_{t \in (0,T)})$ that is κ-noncollapsed at scales < 1 and $R > 0$.
2. $\mathcal{M}_0$ can be compactified by adding a single point. $\mathcal{M}_0 \cong (M - X, g_0 := \lim_{t \searrow 0} g_t)$ for some $X \subset M$.
3. $\rho_{\min}(t) := \min_M \rho(\cdot, t) = \rho(x_t, t)$ is weakly increasing.
4. Every $(x, t) \in M \times (0, T)$ with $\rho(x, t) < 1$ is one of the following:
    - a center of a strong ε-neck at scale $\rho(x, t)$
    - locally ε-modeled on (M_{Bry}, g_{Bry}, y) for some $y \in M_{Bry}$ with $d_t(y, x_{Bry}) < C_{Bry}(\varepsilon)$.

(ε will be chosen small in the following)

Strategy

- Let $\mathcal{M}$, $\mathcal{M}'$ be as before and suppose $\mathcal{M}_0 \cong \mathcal{M}'_0$. Want to show $\mathcal{M} \cong \mathcal{M}'$.
- If we could show that $\mathcal{M}_t \cong \mathcal{M}'_t$ for some $t > 0$, then $\mathcal{M}_{\geq t} \cong \mathcal{M}'_{\geq t}$.
- Construct $\mathcal{N} \subset \mathcal{M}$ (comparison domain) and
  $\phi : \mathcal{N} \to \mathcal{M}'$ (comparison map) that is $(1+\eta)$-bilipschitz
- Let $\eta \to 0$ and $\mathcal{N} \to \mathcal{M}$, limit $\rightsquigarrow$ isometry between $\mathcal{M}, \mathcal{M}'$



Lecture II.C

Uniqueness in an even more basic case: the non-singular case

Theorem

$$(M, g(0)) \cong (M', g'(0)) \qquad \Longrightarrow \qquad (M, g(t)) \cong (M', g'(t))$$

Proof

- **Comparison map:** $\phi : M \longrightarrow M'$ such that $\phi^* g'(0) = g(0)$
- **Perturbation:** $h(t) = \phi^* g'(t) - g(t), \qquad h(0) \equiv 0$
- **DeTurck's trick:** If ϕ_t^{-1} moves by harmonic map heat flow

  $$\partial_t \phi_t^{-1} = \triangle_{g'(t), g(t)} \phi_t^{-1},$$

  then

  $$\partial_t h = \triangle_{g(t)} h + 2\,\mathrm{Rm}_{g(t)}(h) + \nabla h * \nabla h + h * \nabla^2 h$$

  (Ricci-DeTurck flow)
- Standard parabolic theory: $h(0) \equiv 0 \quad \Longrightarrow \quad h(t) \equiv 0$ **q.e.d.**

Later:

If $|h(t)| < \eta_{\mathrm{lin}} \ll 1$, then

$$\partial_t h \approx \triangle_{g(t)} h + 2\,\mathrm{Rm}_{g(t)}(h) \qquad \text{where} \qquad (\mathrm{Rm}(h))_{ij} = R_{istj} h_{st}$$

(linearized Ricci-DeTurck flow)

Lecture II.D

Local stability analysis

Linearized Ricci DeTurck flow

(g_t) Ricci flow with $R > 0$ (for simplicity)
(h_t) linearized Ricci DeTurck flow: $\partial_t h = \triangle_{g(t)} h + 2\,\mathrm{Rm}_{g(t)}(h)$

Anderson, Chow (2005)

$$\partial_t \frac{|h|}{R} \leq \triangle \frac{|h|}{R} - 2\frac{\nabla R}{R} \cdot \nabla \frac{|h|}{R}$$

So $\max_M \frac{|h|}{R}(\cdot, t)$ is non-increasing in t.

Proof: $\Box = \partial_t - \triangle$

$$\begin{aligned} \Box \frac{|h|}{R} &= \frac{1}{R^2}\left(\Box|h| \cdot R - |h|\Box R\right) - 2\frac{\nabla R}{R} \cdot \nabla \frac{|h|}{R} \\ &\leq \frac{1}{R^2}\left(\frac{2\,\mathrm{Rm}(h,h)}{|h|} \cdot R - |h| \cdot 2|\mathrm{Ric}|^2\right) - 2\frac{\nabla R}{R} \cdot \nabla \frac{|h|}{R} \end{aligned}$$

Need

$$\mathrm{Rm}(h,h)R \leq |h|^2|\mathrm{Ric}|^2$$

Can be checked using an "elementary" computation.

Interpretation

Anderson, Chow (2005)

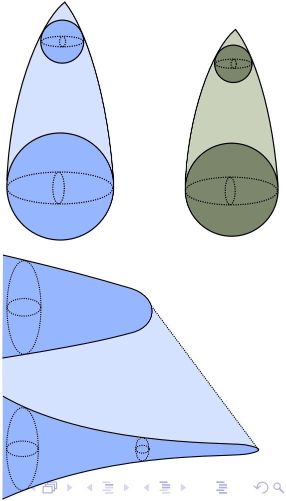
$$\max_M \frac{|h|}{R} \quad \text{is non-increasing}$$

Disadvantage:

- Differences at macroscopic scales ($R \approx 1$) may have big effect at microscopic scales ($R \gg 1$).
- example: $$g(t) = (1-4t)g_{S^3},$$ $$g'(t) = (0.9-4t)g_{S^3},$$ $$|h|_{g(t)}(t) \sim \frac{0.1}{1-4t} \sim R(\cdot, t).$$

Advantage:

- Differences at microscopic scales ($R \gg 1$) have little effect at macroscopic scales ($R \approx 1$).
- example: Ricci flow regularizing infinitesimal surgery.



Vanishing Theorem

Let $(M, (g_t)_{t\in(-\infty,0]})$ be a κ-solution and $(h_t)_{t\in(-\infty,0]}$ a linearized RDTF s.t.

$$|h| \leq CR^{1+\gamma}, \qquad \gamma > 0.$$

Then $h \equiv 0$.

Proof: $|h| \leq CR^{1+\gamma} \leq CC'R$. Suppose that $(x_i, t_i) \in M \times (-\infty, 0]$ s.t.

$$\frac{|h|}{R}(x_i, t_i) \xrightarrow[i\to\infty]{} \sup_{M\times(-\infty,0]} \frac{|h|}{R} =: C_0 > 0.$$

Then

$$R^{\gamma}(x_i, t_i) = \frac{R^{1+\gamma}}{R}(x_i, t_i) \geq \frac{C^{-1}|h|}{R}(x_i, t_i) \xrightarrow[i\to\infty]{} C^{-1}C_0 > 0 \qquad (*)$$

So $R(x_i, t_i) > c > 0$. After passing to a subsequence

$$(M, (g_{t+t_i})_{t\in(-\infty,0]}, x_i) \xrightarrow[i\to\infty]{C^{\infty}-HCG} (M_{\infty}, (g^{\infty}_t)_{t\in(-\infty,0]}, x_{\infty})$$

$$(h_{t+t_i})_{t\in(-\infty,0]} \xrightarrow[i\to\infty]{} (h^{\infty}_t)_{t\in(-\infty,0]}$$

$|h^{\infty}| \leq CR^{1+\gamma}$, $\qquad \dfrac{|h^{\infty}|}{R} \leq C_0$ $\qquad$ with equality at $(x_{\infty}, 0)$

$|h^\infty| \leq CR^{1+\gamma}, \qquad \frac{|h^\infty|}{R} \leq C_0 \qquad$ with equality at $(x_\infty, 0)$

Anderson-Chow + strong maximum principle $\qquad \Longrightarrow \qquad \frac{|h^\infty|}{R} \equiv C_0$

$$R^\gamma = \frac{R^{1+\gamma}}{R} \geq \frac{C^{-1}|h^\infty|}{R} = C^{-1} C_0^{-1} > 0$$

on $M_\infty \times (-\infty, 0]$.
This is false on any κ-solution. q.e.d.

Linear $\longrightarrow$ Non-Linear

- (g_t) RF background
- (h_t) solution to non-linear RDTF

$$\partial_t h = \triangle h + 2\,\mathrm{Rm}(h) + \nabla h * \nabla h + h * \nabla^2 h$$

- **Observation:** Divide by $0 < a \ll 1$

$$\partial_t \left(\frac{h}{a} \right) = \triangle \left(\frac{h}{a} \right) + 2\,\mathrm{Rm}\left(\frac{h}{a} \right) + a \cdot \nabla \left(\frac{h}{a} \right) * \nabla \left(\frac{h}{a} \right) + a \cdot \left(\frac{h}{a} \right) * \nabla^2 \left(\frac{h}{a} \right)$$

- If $a_i \to 0$ and $\dfrac{h_i}{a_i} \to h_\infty$, then (assuming certain derivative bounds)

$$\partial_t h_\infty = \triangle h_\infty + 2\,\mathrm{Rm}(h_\infty)$$

(linearized RDTF)

$(g_t)_{t\in[0,T)}$ κ-noncollapsed RF, ε-canonical nbhd assumption below scale 1,
$(h_t)_{t\in[0,T)}$ (non-linear) RDTF

$$Q := e^{-Ht}\frac{|h|}{R^E}, \qquad E > 1$$

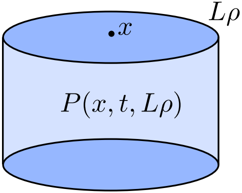
Semi-local maximum principle + Interior decay

For any $\alpha, \kappa > 0$, $E > 1$ there are $H, \varepsilon, \eta, L > 0$ s.t. if

$$|h| \leq \eta \qquad on \qquad P = P(x,t,L\rho(x,t)),$$

then

$$Q(x,t) \leq \alpha \sup_P Q. \qquad (*)$$



Proof: Fix α, κ, E and argue by contradiction. Choose $H_i, L_i \to \infty$, $\eta_i, \varepsilon_i \to 0$.
Counterexamples $(M_i, (g^i_t))$, (h_i), $|h_i| \leq \eta_i \to 0$, $r_i := \rho(x_i, t_i)$.
Opposite of $(*)$: On P_i

$$\alpha e^{-Ht}\frac{|h_i|}{R^E} \leq \alpha \sup_{P_i} Q < Q(x_i,t_i) = e^{-Ht_i}\frac{|h_i|}{R^E}(x_i,t_i)$$

$$\frac{|h_i|}{|h_i|(x_i,t_i)} \leq \alpha^{-1} e^{-H_i(t_i - t)} \frac{R^E}{R^E(x_i,t_i)}$$

$$\frac{|h_i|}{|h_i|(x_i,t_i)} \leq \alpha^{-1} e^{-H_i(t_i-t)} \frac{R^E}{R^E(x_i,t_i)} \qquad \text{on} \qquad P_i$$

After passing to a subsequence ($r_i = \rho(x_i, t_i)$):

$$(M_i, (r_i^{-2} g_{r_i^2 t + t_i}), x_i) \xrightarrow[i \to \infty]{C^\infty - HCG} (M_\infty, (g_t^\infty)_{t \in (-\infty, 0]}, x_\infty)$$

$$\frac{h_i}{|h_i|(x_i,t_i)} \xrightarrow[i \to \infty]{} (h_{\infty,t})_{t \in (-\infty,0]} \qquad \text{(linearized RDTF)}$$

Then $|h_\infty|(x_\infty, 0) = \rho(x_\infty, 0) = 1$

$$|h_\infty|(y,t) \leq \lim_{i \to \infty} 10 \alpha^{-1} e^{H_i r_i^2 \cdot t} R^E(y,t)$$

Case $\liminf_{i \to \infty} \mathbf{r_i} > \mathbf{0}$: $|h_\infty|(\cdot, t) \equiv 0$ for $t < 0$ $\implies$ $h_\infty \equiv 0$

Case $\liminf_{i \to \infty} \mathbf{r_i} = \mathbf{0}$: limit is κ-solution
$|h_\infty| \leq 10\alpha^{-1} R^E$ Vanishing Thm $\implies$ $h_\infty \equiv 0$

q.e.d.

Recap

$(g_t)_{t\in[0,T)}$ κ-noncollapsed RF, ε-canonical nbhd assumption below scale 1,
$(h_t)_{t\in[0,T)}$ (non-linear) RDTF

$$Q := e^{-Ht}\frac{|h|}{R^E}, \qquad E > 1$$

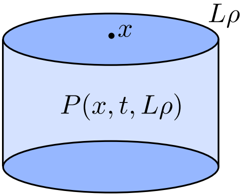
Semi-local maximum principle + Interior decay

For any $\alpha, \kappa > 0$, $E > 1$ there are $H, \varepsilon, \eta, L > 0$ s.t. if

$$|h| \leq \eta \qquad on \qquad P = P(x, t, L\rho(x, t)),$$

then

$$Q(x, t) \leq \alpha \sup_P Q. \qquad (*)$$



Consequences (and minor generalizations):

- If $Q(\cdot, 0) \leq H^{-1}\overline{Q}$ and $|h| \leq \eta$ on $M \times [0, T)$, then $Q \leq \overline{Q}$ on $M \times [0, T)$.
- If $Q(\cdot, 0) \leq H^{-1}\overline{Q}$ and if $\overline{Q}$ is chosen such that

  $$Q \leq \overline{Q} \qquad \Longrightarrow \qquad |h| \leq \eta,$$

  then $Q \leq \overline{Q}$ and $|h| \leq \eta$.
- Can make η independent of α, κ.

Lecture III

A Construction of the comparison map

B The Cap Extension

C The general case

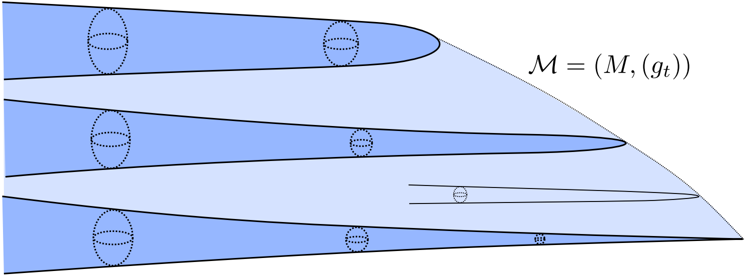
D Precise statements of the results + further byproducts of the proof

E Continuous Dependence

Lecture III.A

Construction of the comparison map

Recap: Toy case



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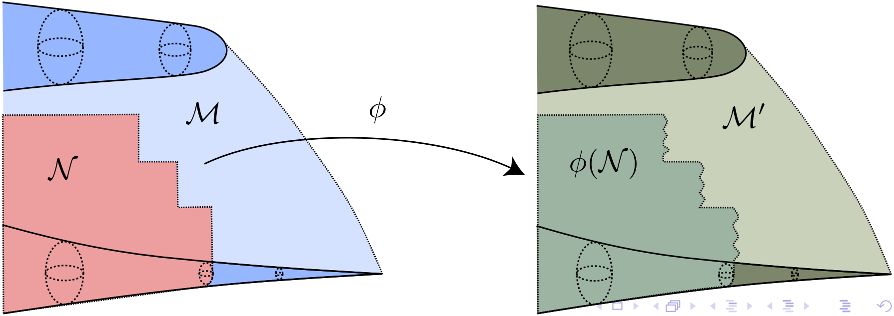
2. $\mathcal{M}_0$ can be compactified by adding a single point. $\mathcal{M}_0 \cong (M - X, g_0 := \lim_{t \searrow 0} g_t)$ for some $X \subset M$.
3. $\rho_{\min}(t) := \min_M \rho(\cdot, t) = \rho(x_t, t)$ is weakly increasing.
4. Every $(x,t) \in M \times (0,T)$ with $\rho(x,t) < 1$ is one of the following:
    - a center of a strong ε-neck at scale $\rho(x,t)$
    - locally ε-modeled on (M_{Bry}, g_{Bry}, y) for some $y \in M_{Bry}$ with $d_t(y, x_{Bry}) < C_{Bry}(\varepsilon)$.

Recap: Strategy

- Let $\mathcal{M}$, $\mathcal{M}'$ be as in toy case and s.t. $\mathcal{M}_0 \cong \mathcal{M}'_0$. Want to show $\mathcal{M} \cong \mathcal{M}'$.
- Construct $\mathcal{N} \subset \mathcal{M}$ (comparison domain) and

  $\phi : \mathcal{N} \to \mathcal{M}'$ (comparison map) that is $(1+\eta)$-bilipschitz

  and such that ϕ^{-1} locally evolves by harmonic map heat flow
- Let $\eta \to 0$ and $\mathcal{N} \to \mathcal{M}$, limit $\rightsquigarrow$ isometry between $\mathcal{M}, \mathcal{M}'$
- Set $h := \phi^* g' - g$ (evolves by non-linear RDTF).
- h can be controlled via semi-local maximum principle far from $\partial\mathcal{N}$



Construction of the comparison domain

- $\delta_n > 0$ and $\lambda = \lambda(\delta_n)$ to be determined later
- Choose $r_{\mathrm{comp}} \ll 1$ (comparison scale)
- $t_j := j \cdot r^2_{\mathrm{comp}}$, $j = 0, \ldots, J$
- $t_J = T$
- choose $j_0 \in \{0, \ldots, J\}$ minimal s.t. $\rho_{\min}(t_{j_0}) = \min_M \rho(\cdot, t_{j_0}) \geq \lambda r_{\mathrm{comp}}$

Lemma

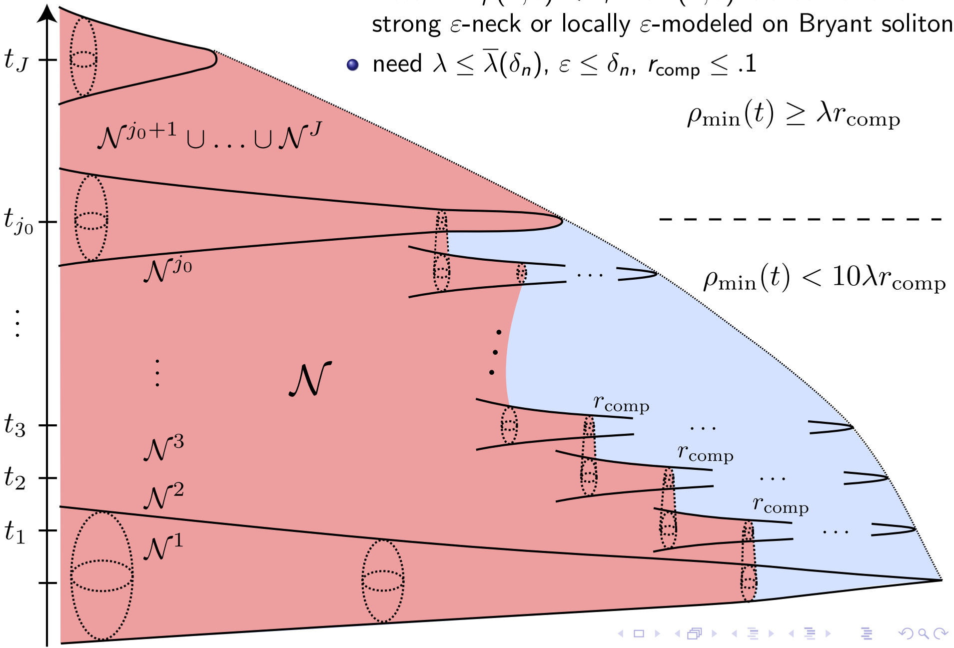
There is a comparison domain

$$\mathcal{N} = (\mathcal{N}^1 \cup \ldots \cup \mathcal{N}^{j_0}) \cup (\mathcal{N}^{j_0+1} \cup \ldots \cup \mathcal{N}^J) \quad \subset \quad \mathcal{M}$$

such that

- $\mathcal{N}^j = N_j \times [t_{j-1}, t_j], \quad M \supset N_1 \supset N_2 \supset \ldots \supset N_{j_0}$
- $\partial \mathcal{N}^j_{t_j}$ is central 2-sphere of strong δ_n-neck at scale r_{comp}
- $\rho > \frac{1}{2} r_{\mathrm{comp}}$ on $\mathcal{N}^1 \cup \ldots \cup \mathcal{N}^{j_0}$
- $\mathcal{N}^j = M \times [t_{j-1}, t_j]$ for $j \geq j_0 + 1$

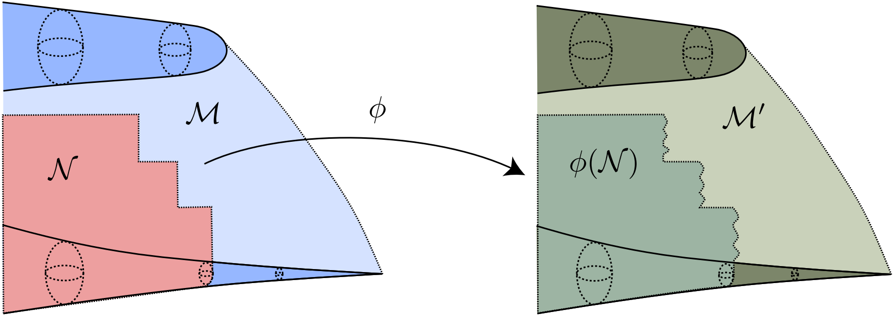
- $\partial \mathcal{N}^j_{t_j}$ must be central 2-sphere of strong δ_n-neck
- Recall: If $\rho(x,t) < 1$, then (x,t) is a center of a strong ε-neck or locally ε-modeled on Bryant soliton
- need $\lambda \leq \overline{\lambda}(\delta_n)$, $\varepsilon \leq \delta_n$, $r_{\text{comp}} \leq .1$



Construction of comparison map for $t \leq t_{j_0}$

Goal: Construct comparison map $\phi : \mathcal{N}^1 \cup \ldots \cup \mathcal{N}^{j_0} \to \mathcal{M}'$, s.t. ϕ^{-1} evolves by harmonic map heat flow s.t.

$$|h| = |\phi^* g' - g| \leq \eta$$



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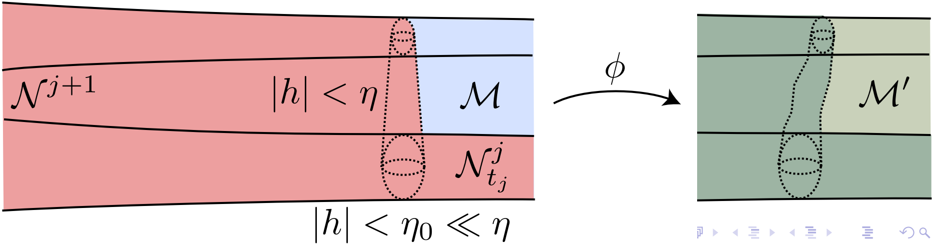
$$|h| = |\phi^* g' - g| \leq \eta$$

Strategy:

- $\phi|_{\mathcal{N}_0}$ given since $\mathcal{M}_0 \cong \mathcal{M}'_0$
- For each $j = 0, \ldots, j_0 - 1$ solve HMHF with initial data $(\phi|_{\mathcal{N}^j_{t_j}})^{-1}$ (graft half cylinders into $\partial \mathcal{N}^{j+1}$ and its image) $\rightsquigarrow$ $\phi|_{\mathcal{N}^{j+1}} : \mathcal{N}^{j+1} \to \mathcal{M}'$.
- If $\delta_n \leq \overline{\delta}_n(\eta')$ and $|h| < \eta_0(\eta') \ll \eta'$ near $\partial \mathcal{N}^{j+1}_{t_j}$, then $|h| < \eta'$ near $\partial \mathcal{N}^{j+1}$.



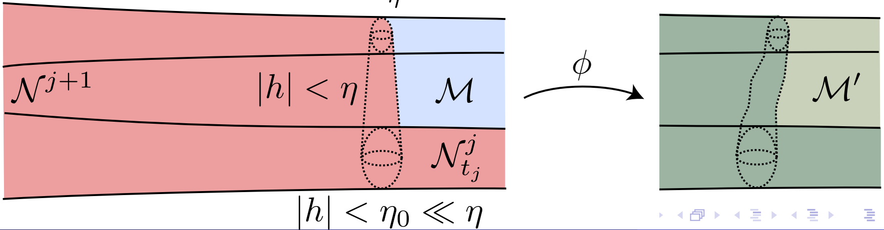
So we need to ensure that :

1. $|h| < \eta_0(\eta') \ll \eta'$ near $\partial\mathcal{N}^{j+1}_{t_j}$ for some $\eta' \leq \eta$
2. $|h| \leq \eta$ on $\mathcal{N}^{j+1}$

Strategy:

- Fix $E \approx 100 > 1$ (TBD) and $Q = e^{-Ht}\frac{|h|}{R^E}$
- Choose $\overline{Q} \approx e^{-HT}\eta r^{2E}_{\text{comp}}$ such that
  - On $\mathcal{N}^{j+1}$: $Q \leq \overline{Q} \quad \Longrightarrow \quad |h| < \eta$
  - Near $\partial\mathcal{N}^{j+1}$: $|h| < \frac{e^{-HT}}{100}\eta =: \eta' \quad \Longrightarrow \quad Q \leq \overline{Q}$
- $Q \leq \overline{Q}$ can be maintained via semi-local maximum principle.
- Moreover: If $d_{t_j}(\partial\mathcal{N}^{j+1}_{t_j}, \partial\mathcal{N}^{j}_{t_j}) > L(\frac{\eta_0}{\eta})r_{\text{comp}}$, then near $\partial\mathcal{N}^{j+1}_{t_j}$:

$$Q < \frac{\eta_0}{\eta}\overline{Q} \qquad \Longrightarrow \qquad |h| < \eta_0$$



Recap: Choice of constants

- Ensure that $\partial\mathcal{N}$ consists of strong δ_n-necks: $\lambda \leq \overline{\lambda}(\delta_n)$, $\varepsilon \leq \delta_n$, $r_{\text{comp}} \leq .1$
- Ensure that $|h| < \eta'$ near $\partial\mathcal{N}^{j+1}$: $\delta_n \leq \overline{\delta}_n(\eta')$, $\eta_0 \leq \overline{\eta}_0(\eta')$
- Apply semi-local maximum principle ($Q \leq \overline{Q}$): $\eta \leq \overline{\eta}$ and $\eta' = \frac{e^{-HT}}{100}\eta$
- Ensure that $d_{t_j}(\partial\mathcal{N}^{j+1}_{t_j}, \partial\mathcal{N}^{j}_{t_j}) > L(\frac{\eta_0}{\eta}) r_{\text{comp}}$ (to show $|h| < \eta_0$ near $\mathcal{N}^{j+1}_{t_j}$): $\delta_n \leq \overline{\delta}_n(\frac{\eta_0}{\eta})$

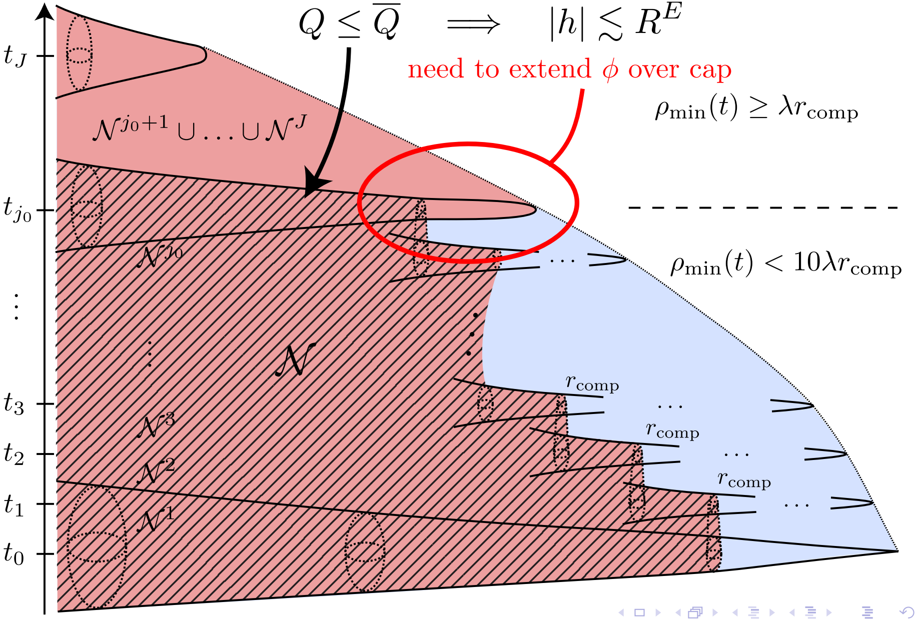
Summary:

$$\begin{tikzcd}
\eta \arrow[r] & \eta' \arrow[r] & \eta_0 \arrow[r] & \delta_n \arrow[r] \arrow[d] & \lambda \\
& & & \varepsilon &
\end{tikzcd}$$

Lecture III.B

The Cap Extension

Construction of comparison map at $t = t_{j_0}$



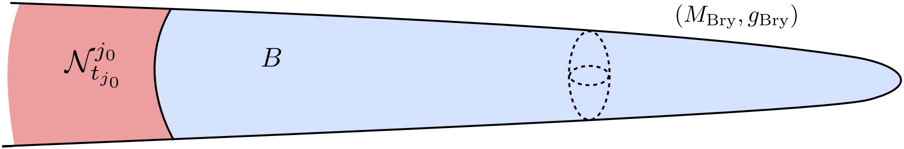
$Q \leq \overline{Q} \quad \Longrightarrow \quad |h| \lesssim R^E$

need to extend ϕ over cap

$\rho_{\min}(t) \geq \lambda r_{\text{comp}}$

$\rho_{\min}(t) < 10\lambda r_{\text{comp}}$

Cap Extension Problem



- diameter of B $\approx \lambda^{-1} r_{\mathrm{comp}}$
- Precision of ϕ near $\partial \mathcal{N}^{j_0}_{t_{j_0}}$ $\approx \eta_0$
- In order to construct an extension over the cap of precision $\approx \eta$ we would need to have $\eta_0 \leq \overline{\eta}_0(\eta, \lambda)$
- But we have chosen λ depending on $\frac{\eta_0}{\eta}$.

Idea:

- Near $\partial \mathcal{N}^{j_0}_{t_{j_0}}$ we have $Q \leq \overline{Q} \quad \Longrightarrow \quad |h| \lesssim \eta (R r_{\mathrm{comp}}^2)^E$
- ϕ has much better precision further away from cap
- Extend over larger domain!

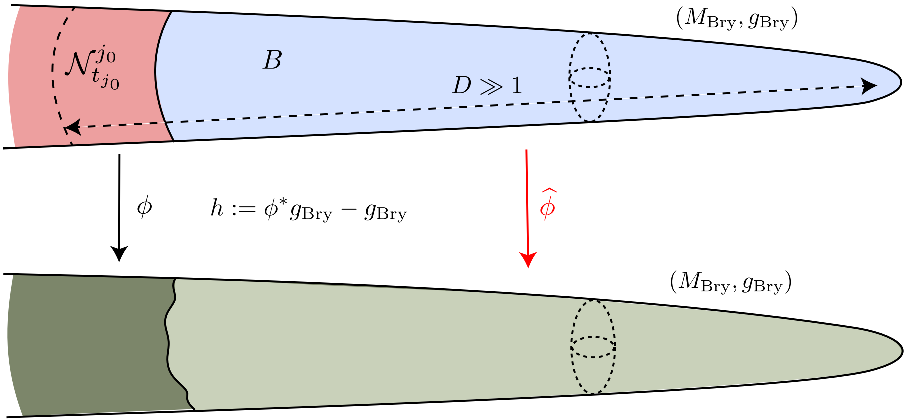
Cap Extension

Cantilever Paradox

Where do you feel safer?

long cantilever,
good engineering

short cantilever,
sketchy engineering



Rescale s.t. geometry around $B := \mathcal{M}_{t_{j_0}} - \mathcal{N}^{j_0}_{t_{j_0}}$ is close to $(M_{\mathrm{Bry}}, g_{\mathrm{Bry}})$ at scale 1.

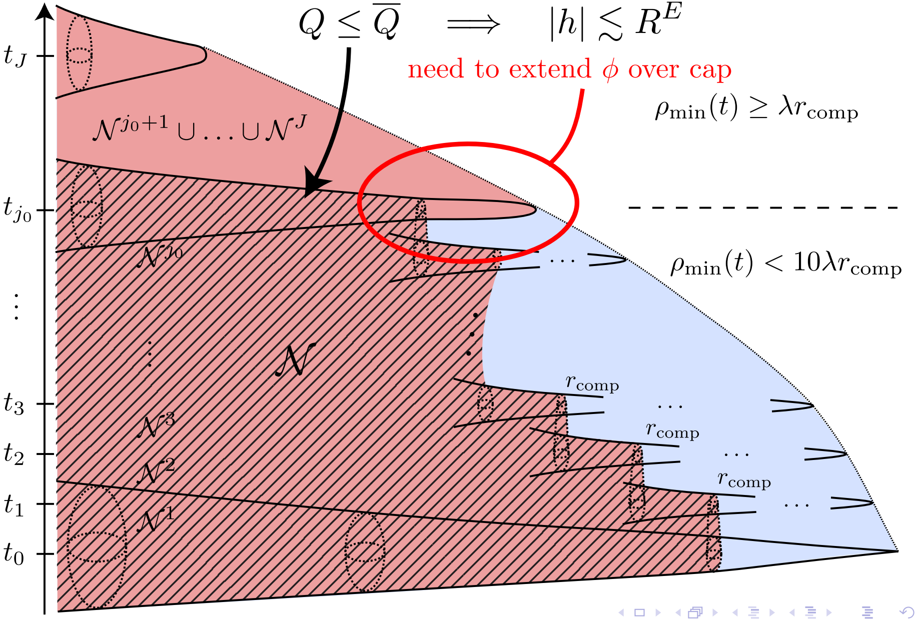
Recall that $Q = e^{-Ht} \dfrac{|h|}{R^E} \leq \overline{Q}$ on $\mathcal{N}$.

Bryant Extension Principle ("Cap Extension")

If $E \geq 100$ and $D \geq D_0(\beta)$, then we can extend $\phi|_{\mathcal{N}^{j_0}_{t_{j_0}} - B(x_{\mathrm{Bry}}, D)}$ to $\widehat{\phi} : \mathcal{M}_{t_{j_0}} \to M_{\mathrm{Bry}}$ such that

$$\left| \widehat{\phi}^* g_{\mathrm{Bry}} - g_{\mathrm{Bry}} \right| < \beta R^2$$

Proof, concluded



$Q \leq \overline{Q} \implies |h| \lesssim R^E$

need to extend ϕ over cap

$\rho_{\min}(t) \geq \lambda r_{\mathrm{comp}}$

$\rho_{\min}(t) < 10\lambda r_{\mathrm{comp}}$

- Extend $\phi|_{\mathcal{N}^{j_0}_{t_{j_0}}}$ onto $\mathcal{N}^{j_0+1}_{t_{j_0}}$ via Bryant Extension Principe (at time t_{j_0}).
- Then at time t_{j_0}:
    - $Q \leq \overline{Q}$ on complement of extension disk
    - $|h| \leq \beta R^2 (\lambda r_{\text{comp}})^4$ on extension disk
- $\Longrightarrow$ on $\mathcal{M}_{t_{j_0}}$:
    - $Q \leq W\overline{Q}$ for some $W(\lambda) \gg 1$
    - $Q^* = e^{-H^* t} \frac{|h|}{R^2} \leq \overline{Q}^* \approx e^{-H^* T} \eta (\lambda r_{\text{comp}})^4$
- The bound $Q^* \leq \overline{Q}^*$ can be maintained using the semi-local maximum principle (exponent $E^* = 2$) and implies that $|h| \leq \eta$ for times $t \geq t_{j_0}$.
- (By the semi-local maximum principle, the bound $Q \leq W\overline{Q}$ eventually improves to $Q \leq \overline{Q}$.)
- q.e.d.

Lecture III.C

The general case

The general case

Problems that may arise

- Caps may not always be modeled on Bryant solitons (if we don't assume classification of κ-solutions)
  Solution: Perform cap extension "at the right time",
  when $\partial_t R \leq 0$ on $\mathcal{M}$ and $\mathcal{M}'$
- There may be more than one cusp
  Solution: Continue the neck and cap extension process after the first cap extension.
- Curvature of the cap may increase again after cap extension (and there may be an accumulation of singular times)
  Solution: Perform a "cap removal" when the curvature exceeds a certain threshold.
  Ensure that "cap extensions" and "cap removals" are sufficiently separated in space and time such that Q has time to "recover" after a cap extension.

Lecture III.D

Precise statements of the results + further byproducts of the proof

Uniqueness Theorem

Uniqueness

If $\mathcal{M}, \mathcal{M}'$ are singular Ricci flows and $\phi : (\mathcal{M}_0, g_0) \to (\mathcal{M}'_0, g'_0)$ is an isometry, then there is a unique extension $\widehat{\phi} : \mathcal{M} \to \mathcal{M}'$, $\widehat{\phi}|_{\mathcal{M}_0} = \phi$, which is an isometry of Ricci flow spacetimes.

Note:

- $\mathcal{M}$, $\mathcal{M}'$ could be non-compact or start from singular initial data (need to modify definition of completeness slightly).

The most general consequence

Strong Stability Theorem

If $\delta > 0$, $T < \infty$, $E \geq \underline{E}$, then there are $\varepsilon_{\mathrm{can}}(E), \varepsilon(\delta, T, E) > 0$ such that for all $r \in (0, 1)$ the following holds:

Suppose $\mathcal{M}, \mathcal{M}'$ are Ricci flow spacetimes over $[0, T)$ that are εr-complete and satisfy the $\varepsilon_{\mathrm{can}}$-canonical neighborhood assumption at scales $(\varepsilon r, 1)$.

Let $\phi : \mathcal{M}_0 \supset U \longrightarrow U' \subset \mathcal{M}'_0$ be a diffeomorphism such that:

- $U \supset \{|\mathrm{Rm}| \leq (\varepsilon r)^{-2}\}$
- $|\phi^* g'_0 - g_0| \leq \varepsilon r^{2E}(|\mathrm{Rm}| + 1)^E$ on U
- "ϕ does not distort $|\mathrm{Rm}|$ too much" ...

Then there is a time-preserving diffeomorphism

$$\widehat{\phi} : \mathcal{M} \supset \widehat{U} \longrightarrow \widehat{U}' \subset \mathcal{M}'$$

such that:

- $\widehat{U} \supset \{|\mathrm{Rm}| \leq r^{-2}\}$
- $\widehat{\phi}$ evolves by HMHF
- $|\widehat{\phi}^* g' - g| \leq \delta r^{2E}(|\mathrm{Rm}| + 1)^E$
- $\widehat{\phi}_0 = \phi$ on $U \cap \widehat{U}$

Note:

- $\mathcal{M}$, $\mathcal{M}$ only need to be defined above scale εr or they may violate the RF equation below that scale.

Consequences:

Convergence of RF with surgery

Fix some closed (M^3, g). Consider a sequence of RFs with surgery $\mathcal{M}^{\delta_i}$ starting from (M, g), for a sequence of surgery scales $\delta_i \to 0$. Then (without passing to a subsequence) we have convergence $\mathcal{M}^{\delta_i} \to \mathcal{M}^\infty$, where $\mathcal{M}^\infty$ is the singular RF starting from (M, g).

Weak Stability Theorem

For every δ, $T > 0$ there is an $\varepsilon(\delta, T) > 0$ such that

If $\mathcal{M}, \mathcal{M}'$ are singular RFs and there is a $(1+\varepsilon)$-bilipschitz map $\phi : \mathcal{M}_0 \to \mathcal{M}'_0$, then there is a $(1+\delta)$-bilipschitz map

$$\widehat{\phi} : \mathcal{M} \supset \widehat{U} \longrightarrow \widehat{U}' \subset \mathcal{M}'$$

such that $\{|\mathrm{Rm}| \leq \delta^{-2}\} \subset \widehat{U}$ and $\widehat{\phi} = \phi$ on $\widehat{U} \cap U$.

Convergence

Consider a sequence of singular Ricci flows $\mathcal{M}^i$ such that there is a sequence of diffeomorphisms $\phi_i : \mathcal{M}^\infty_0 \to \mathcal{M}^i_0$ with

$$\phi_i^* g_0^i \xrightarrow[i\to\infty]{C^\infty_{\mathrm{loc}}} g_0^\infty .$$

Then there is a sequence of time-preserving diffeomorphisms

$$\widehat{\phi}_i : \mathcal{M}^\infty \supset \widehat{U}_i \longrightarrow \widehat{U}'_i \subset \mathcal{M}^i$$

such that $\widehat{U}^i \nearrow \mathcal{M}^\infty$ and

$$\widehat{\phi}_i^* g^i \xrightarrow[i\to\infty]{C^\infty_{\mathrm{loc}}} g^\infty, \qquad \widehat{\phi}_i^* \partial_{\mathfrak{t}}^i \xrightarrow[i\to\infty]{C^\infty_{\mathrm{loc}}} \partial_{\mathfrak{t}}^\infty$$

Question: How to state continuous dependence on initial data?

1. Quick and dirty method (next two slides)
2. More comprehensive method (later)

Lecture III.E

Continuous dependence

Worldlines and bad points

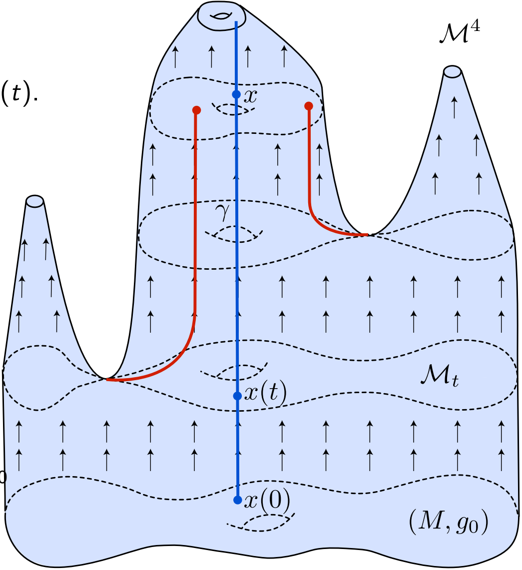
Worldline: Trajectories γ of ∂_t.

If $\gamma(t_0) = x$, then we write $x(t) := \gamma(t)$.

A point $x \in M$ is called **bad point**
if $x(0)$ does not exist
(i.e. worldline is incomplete).

Lemma (Kleiner, Lott)

There are at most finitely many bad points in every component $\mathcal{C} \subset \mathcal{M}_t$ of every time-slice.



$W_t := \{x(0) \mid x \in \mathcal{M}_t \text{ good}\} \subset \mathcal{M}_0$
$\overline{g}_t :=$ pushforward of g_t onto W_t
by flow of $-\partial_t$

Note: $(W_t, \overline{g}_t) \cong (\mathcal{M}_t, g_t) -$ bad points

Continuous dependence — quick and dirty

Recall:

singular Ricci flow $\mathcal{M}$

$$\rightsquigarrow \quad (W_t, \overline{g}_t)_{t\geq 0}, \quad \text{where } W_t \subset \mathcal{M}_0 \text{ open, decreasing, } (W_0, \overline{g}_0) = (\mathcal{M}_0, g_0) \text{ and } (\overline{g}_t) \text{ satisfies the Ricci flow equation}$$

Continuous dependence

Fix M^3 closed. For any Riemannian metric g on M consider a singular Ricci flow $\mathcal{M}^{(M,g)}$ with initial time-slice (M, g). Then the map

$$g \longmapsto \mathcal{M}^{(M,g)} \longmapsto (W_t, \overline{g}_t)_{t\geq 0}$$

is continuous in the C^∞_{loc}-topology. More specifically, if $g^i \to g^\infty$ in C^∞_{loc}, then for any $t \geq 0$ we have:

- $W^\infty_t \subset \liminf_{i\to\infty} W^i_t$, i.e. for any compact $K \subset W^\infty_t$ we have $K \subset W^i_t$ for large i.
- $\overline{g}^i_t \to \overline{g}^\infty_t$ in C^∞_{loc} on W^∞_t.

Note: It may happen that $W^i_t = M$, but $W^\infty_t \subsetneq M$.

Lecture IV

A Short/Partial Proof of the Generalized Smale Conjecture
B Continuous families of singular Ricci flows
C (Long) Proof of both topological conjectures

Lecture IV.A

Short/Partial Proof of the Generalized Smale Conjecture

Setup

Assume $M = S^3/\Gamma$, where $\Gamma \neq 1, \mathbb{Z}_2$ and assume $\mathrm{Diff}(S^3) \simeq O(4)$ (Hatcher)

Goal:

Theorem

$$\mathrm{Met}_{K\equiv 1}(M) \text{ is contractible}$$

$$(\text{consequently } \mathrm{Diff}(M) \simeq \mathrm{Isom}(M))$$

"Proof":

- **Hope:** Construct retraction

$$* \simeq \mathrm{Met}(M) \longrightarrow \mathrm{Met}_{K\equiv 1}(M)$$

- **Strategy:**

$$\mathrm{Met}(M) \longrightarrow \{\text{“space of singular RFs”}\} \longrightarrow \mathrm{Met}_{K\equiv 1}(M)$$

- **Question** How to construct the second map?

There are $T_g^1 < T_g^2$ such that:

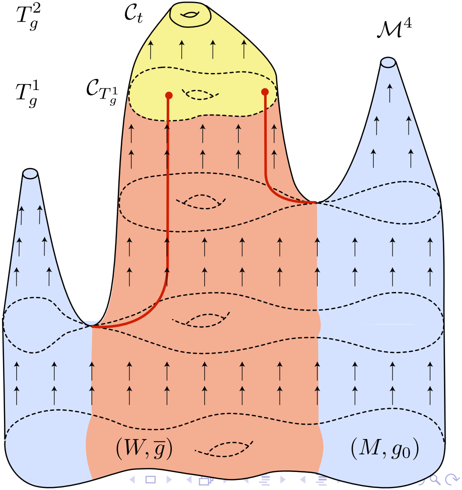
- for every $t \in [T_g^1, T_g^2)$ there is a unique component $\mathcal{C}_t \subset \mathcal{M}_t$ with $\mathcal{C}_t \approx M$.
- $(\mathcal{C}_t, g_t)$ converges to a round metric as $t \nearrow T_g^2$ (modulo rescaling).
  (because $M \neq S^3, \mathbb{R}P^3$!!)

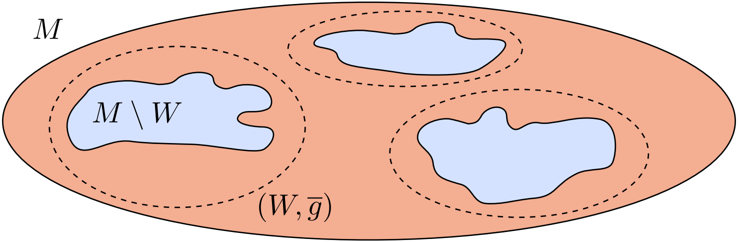
Let $W \subset W_t \subset M$ be the component corresponding to $\mathcal{C}_t$ for $t \in [T_g^1, T_g^2)$.

Recall: $\overline{g}_t :=$ pushforward of g_t onto W by flow of $-\partial_{\mathfrak{t}}$

$\overline{g}_t|_W \xrightarrow[t \nearrow T_g^2]{} \overline{g}$ modulo rescaling $(K_{\overline{g}} \equiv 1)$

$(W, \overline{g}) \cong S^3/\Gamma - \{q_1, \ldots, q_N\}$
$(q_1, \ldots, q_N$ correspond to bad points)





This process describes a continuous, canonical map

$$\begin{array}{ccccc} \mathrm{Met}(M) & \longrightarrow & \{\text{“space of singular RFs”}\} & \longrightarrow & \mathrm{PartMet}_{K\equiv 1}(M) \\ g & \longmapsto & \mathcal{M}^{(M,g)} & \longmapsto & (W,\overline{g}) \end{array}$$

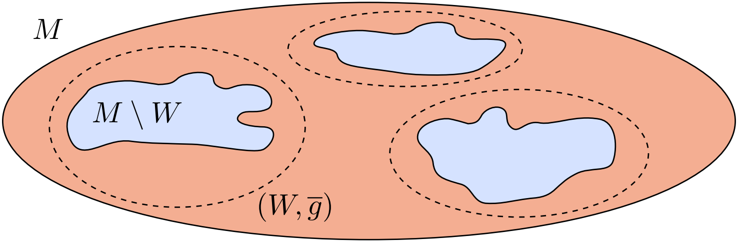
where $\mathrm{PartMet}_{K\equiv 1}(M)$ consists of pairs $(W, \overline{g})$ such that:

- $W \subset M$ open, $\overline{g}$ is a Riemannian metric on W
- $(W, \overline{g})$ is isometric to the round punctured S^3/Γ
- $M - W$ can be covered finitely many, pairwise disjoint disks

If $K_g \equiv 1$, then $(W, \overline{g}) = (M, g)$.

Topology on $\mathrm{PartMet}_{K\equiv 1}(M)$: C^∞_{loc}-convergence on compact subsets of W (not Hausdorff)

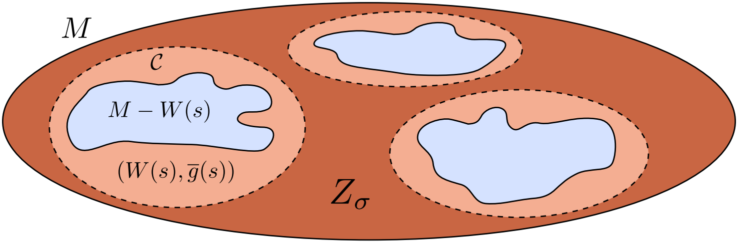
Proof that $\pi_k(\mathrm{Met}_{K\equiv1}(M)) = 0$



- Consider continuous map $g : S^k \to \mathrm{Met}_{K\equiv1}(M)$.
- Extend to $\widetilde{g} : D^{k+1} \to \mathrm{Met}(M)$, $\widetilde{g}|_{S^k=\partial D^{k+1}} = g$.
- Previous slide $\rightsquigarrow$ continuous map $D^{k+1} \to \mathrm{PartMet}_{K\equiv1}(M)$, given by

  $$(W(s), \overline{g}(s)) \in \mathrm{PartMet}_{K\equiv1}(M), \qquad s \in D^{k+1}$$

  such that $(W(s), \overline{g}(s)) = (M, g(s))$ for $s \in \partial D^{k+1}$.
- **Goal:** Replace $\overline{g}(s)$ with a $K \equiv 1$ metric on each disk covering $M - W(s)$, in a continuous fashion.



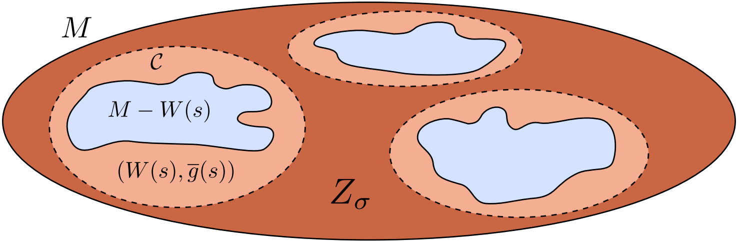
Lemma

We can find a triangulation on D^{k+1} and for each simplex $\sigma \subset D^{k+1}$ a compact subset $Z_\sigma \subset M$ such that the following holds:

1. $Z_\sigma \subset W(s)$ for all $s \in \sigma$
2. $M \setminus \operatorname{Int} Z_\sigma$ consists of finitely many disks
3. If $\sigma_1 \subsetneq \sigma_2$, then $Z_{\sigma_2} \subset \operatorname{Int}(Z_{\sigma_1})$.
4. $Z_\sigma = M$ if $\sigma \subset \partial D^{k+1}$

Proof

- Recall that $(W(s_\sigma), \overline{g}(s_\sigma)) \cong S^3/\Gamma - \{q_1, \ldots, q_N\}$ for some (fixed) $s_\sigma \in \sigma$
- Let Z_σ correspond to S^3/Γ minus subcollection of $\{B(q_i, r_{\dim \sigma})\}$, where $r_j = 10^j r_0 \ll 1$



Proceed by induction over $\dim \sigma$:

- **Goal:** Find a continuous family of $K \equiv 1$ metrics $(\widehat{g}_\sigma(s))_{s \in \sigma}$ that extend $\overline{g}(s)$ over the missing disks of Z_σ.
- Fix $\sigma \subset D^{k+1}$ and disk $\mathcal{C} \subset M \setminus \operatorname{Int} Z_\sigma$.
- Conditions on $\widehat{g}_\sigma(s)|_{\mathcal{C}}$:

① If $s \in \partial\sigma$: $\widehat{g}_\sigma(s)|_{\mathcal{C}} = \widehat{g}_{\sigma'}(s)|_{\mathcal{C}}$ for $s \in \sigma' \subset \partial\sigma$

② Near $\partial\mathcal{C}$: $\widehat{g}_\sigma(s) = \overline{g}(s)$

Main tool (for $\sigma \approx D^{j+1}$)

Lemma

Let $A = A(1-\varepsilon, 1) \subset D(1) \subset \mathbb{R}^3$ and

$$h : D^{j+1} \longrightarrow \mathrm{Met}_{K\equiv 1}(A),$$

$$h_0 : \partial D^{j+1} \longrightarrow \mathrm{Met}_{K\equiv 1}(D(1))$$

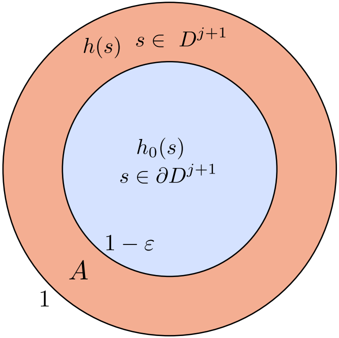
be continuous such that:

- $h_0(s)|_A = h(s)$ for all $s \in \partial D^{j+1}$.
- $(A, h(s))$ embeds into the round sphere for all $s \in D^{j+1}$.

Then, after shrinking ε, there is a continuous map

$$\overline{h} : D^{j+1} \longrightarrow \mathrm{Met}_{K\equiv 1}(D(1))$$

with $\overline{h}(s)|_A = h(s)$ for all $s \in D^{j+1}$ and $\overline{h}(s) = h_0(s)$ for all $s \in \partial D^{j+1}$.



Proof: $\mathrm{Diff}(S^3) \simeq O(3)$ (Hatcher's Theorem) $\Longrightarrow \mathrm{Diff}(D^3 rel \partial D^3) \simeq 1$

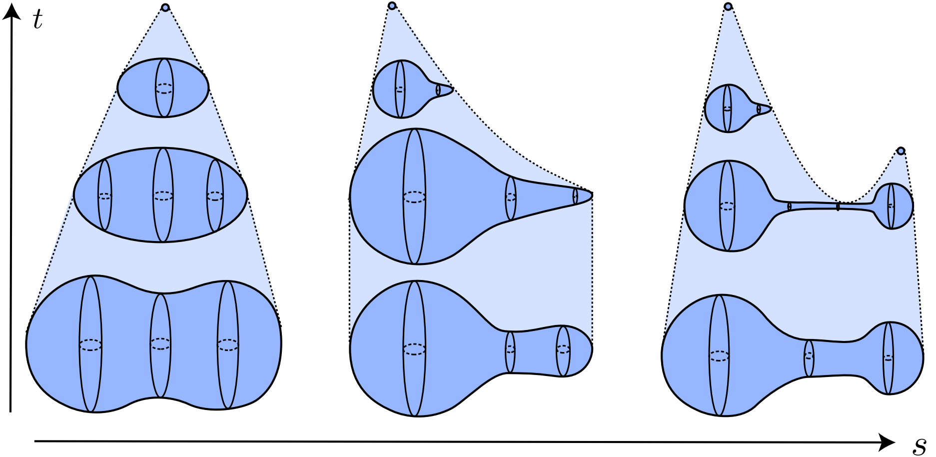
Lecture IV.B

Continuous families of singular Ricci flows

Continuity of RF space-times

continuous family of metrics $(g^s)_{s\in[0,1]}$ on M

$\rightsquigarrow \quad \{\mathcal{M}^s\}_{s\in[0,1]}$ singular Ricci flows



Continuous families of (in-complete) Riemannian manifolds

Note: For fixed t, the family $(\mathcal{M}^s_t, g^s_t)$ is "continuous",
but topology may change

Example: $\pi : \mathbb{R}^3 - \{0\} \longrightarrow \mathbb{R}, \qquad (x, y, z) \mapsto z$

$$M_s := \pi^{-1}(s) = \begin{cases} \mathbb{R}^2 - \{0\} & \text{if } s = 0 \\ \mathbb{R}^2 & \text{else} \end{cases} \qquad g_s := \text{Euclidean metric on } M_s$$

Definition

A continuous family of manifolds over a topological space X is given by:

1. A collection of smooth manifolds $\{M^n_s\}_{s \in X}$
2. A topology on $Y := \bigsqcup_{s \in X} M_s$ that restricts to the topology on each $M_s \subset Y$, such that the standard projection $\pi : Y \to X$ is a continuous topological submersion.
3. A maximal collection of charts

$$\{\phi_{i,s} : V_i \longrightarrow M_s\}_{s \in U_i}, \qquad U_i \subset X \text{ open}, \quad V_i = \text{smooth manifold}$$

such that:

- Each $\phi_i : U_i \times V_i \to Y$ is a homeomorphism onto its open image
- Compatibility condition:

$$\phi^{-1}_{j,s} \circ \phi_{i,s} : V_i \supset V'_{ij,s} \longrightarrow V_j$$

consists of smooth maps that depend continuously on s in the C^∞_{loc}-topology for all i.

A family of Riemannian metrics $\{g_s\}_{s \in X}$ is called transversely continuous if $\phi^*_{i,s} g_s$ depends continuously on s in the C^∞_{loc}-topology.

Definition

A continuous family of Ricci flow spacetimes is given by a continuous family of spacetime manifolds $\{\mathcal{M}^s\}_{s\in X}$ such that $g^s, \mathfrak{t}^s, \partial_{\mathfrak{t}}^s$ are transversely continuous.

Continuity of singular Ricci flows

Let M^3 be closed and $(g^s)_{s\in X}$ be a family or Riemannian metrics that depends continuously on s in the C^∞_{loc}-topology. Choose singular Ricci flows $\mathcal{M}^s$ starting from (M, g^s) for each $s \in X$.
Then $\{\mathcal{M}^s\}_{s\in X}$ can be equipped with a unique structure of a continuous family of Ricci flow spacetimes with the property that the family $\{\mathcal{M}^s_0\}_{s\in X}$ is the trivial family given by $M \times X$.

Note: For any $t \geq 0$, this induces a structure of continuous family of 3-manifolds on $\{\mathcal{M}^s_t\}_{s\in X}$ and the family $\{g^s_t\}_{s\in X}$ is transversely continuous.

Lecture IV.C

(Long) Proof of both topological conjectures

Rounding singular Ricci flows

Observation: If $\mathcal{M}$ is a singular Ricci flow and $\rho(x) \ll 1$, then x is locally ε-modeled on the final time-slice of a κ-solution, which is either round or rotationally symmetric.

Rounding Lemma

Given a continuous family of singular Ricci flows $\{\mathcal{M}^s\}_{s \in X}$ and some $\delta > 0$, there are transversely continuous tensor fields $\{g'^{,s}\}_{s \in X}$, $\{\partial'^{,s}_{\mathfrak{t}}\}_{s \in X}$ such that the family

$$\mathcal{M}'^{,s} = (\mathcal{M}^s, \mathfrak{t}^s, g'^{,s}, \partial'^{,s}_{\mathfrak{t}})$$

forms a continuous family of "rounded, singular almost Ricci flows":

- All properties of singular Ricci flows hold, except for the precise Ricci flow equation, which is replaced by an almost Ricci flow equation.
- $|g^s - g'^{,s}|, |\partial^s_{\mathfrak{t}} - \partial'^{,s}_{\mathfrak{t}}| < \delta$
- At small curvature scales, $g'^{,s}$ is locally **precisely** round or rotationally symmetric and the flow of $\partial'^{,s}_{\mathfrak{t}}$ preserves these symmetries.

Proof: Technical . . .

Proof Strategy

Theorem (Ba., Kleiner 2019)

$\mathrm{Met}_{PSC}(M)$ and $\mathrm{Met}_{K\equiv 1}(M)$ are either contractible or empty.

Recall from 2D proof ($\pi_k(\mathrm{Met}_{\ldots}(M)) = 0$):

- Given $(h_{s,0})_{s\in D^{k+1}}$, PSC or $K \equiv 1$ for $s \in \partial D^{k+1}$
- Need to extend to homotopy $(h_{s,t})_{s\in D^{k+1}\times[0,1]}$ that is PSC or $K \equiv 1$ for $(s,t) \in (\partial D^{k+1} \times [0,1]) \cup (D^{k+1} \times \{1\})$.
- This time: $(h_{s,0})_{s\in D^{k+1}} \rightsquigarrow$ cont. family of rounded, singular almost RFs $(\mathcal{M}'^{,s})_{s\in D^{k+1}}$

Remaining conversion problem: Given a continuous family of rounded sing. RFs $\{\mathcal{M}'^{,s}\}_{s\in X}$, find a "similar" continuous family of metrics $(h_{s,t})_{s\in X, t\in[0,1]}$ s.t.:

1. $(M, h_{s,0}) \cong (\mathcal{M}'^{,s}_0, g^s_0)$.
2. $h_{s,1}$ has $K \equiv 1$ (or PSC).
3. If $\mathcal{M}'^{,s}$ has $K \equiv \mathit{const}$ (or PSC), then so does $h_{s,t}$ for all $t \in [0,1]$.

Condition ② can be reduced to: $h_{s,1}$ is conformally flat

Conversion Problem: Given $\{\mathcal{M}'^{,s}\}_{s\in X}$, find continuous $(h_{s,t})_{s\in X, t\in[0,1]}$ on M:

❶ $(M, h_{s,0}) \cong (\mathcal{M}'^{,s}_0, g^s_0)$.

❷ $h_{s,1}$ is conformally flat.

❸ If $\mathcal{M}'^{,s}$ has $K \equiv const$, then so does $h_{s,t}$ for all $t \in [0,1]$.

"Proof by backwards induction over time:"

- Fix $T \geq 0$ (large)
- Solve conversion problem such that ❷, ❸ hold and instead of ❶:

$$(M, h_{s,0}) \cong (\mathcal{M}'^{,s}_T, g^s_T) \qquad (*_T)$$

- Deform $(h_{s,t})$ such that $(*_{T-\tau})$, ❷, ❸ hold.
- Repeat until $T = 0$.

Problem: The topology of $\mathcal{M}'^{,s}_T$ may change as we vary s.

"Partial homotopy at time T":

- Notion simulating $(*_T)$, ❷, ❸.
- If $T \gg 0$, then $\exists$ empty partial homotopy at time T.
- **Construction moves:** partial homotopy at time T $\rightsquigarrow$ partial homotopy at time $T - \tau$
- A partial homotopy at time $T = 0$ induces the desired family $(h_{s,t})$ with ❶-❸

Partial homotopy at time T (simplified definition):
Fix a simplicial decomposition of X. For each simplex $\sigma \subset X$ choose:

- a continuous family of cpt domains $(Z^\sigma_s \subset \mathcal{M}'^{,s}_T)_{s\in\sigma}$.
- a continuous family of Riemannian metrics $(h^\sigma_{s,t})_{s\in\sigma, t\in[0,1]}$ on (Z^σ_s).

such that $(*_T)$, ❷, ❸ hold and:

- **Compatibility:** If $s \in \tau \subset \sigma$, then $Z^\sigma_s \subset Z^\tau_s$ and $h^\tau_{s,t} = h^\sigma_{s,t}$ on Z^σ_s.
- **Largeness of the domains:** $\rho \ll 1$ on $\mathcal{M}^s_T - Z^\sigma_s$ ($\Rightarrow$ round or rot. symmetric)
- **"Contractible ambiguity":** $h^\tau_{s,t}$ is round or rotationally symmetric on any complement of the form $Z^\tau_s \setminus Z^\sigma_s$.

- If $T \gg 0$, then we can choose $Z^\sigma_s = \emptyset$ (trivial partial homotopy).
- If $T = 0$, $Z^\sigma_s = \mathcal{M}^s_0$, then part. homotopy induces the desired $(h_{s,t})_{s\in X, t\in[0,1]}$

Modification Moves:

- Passing to a simplicial refinement.
- Enlarging some (Z^σ_s) by a family of round or rot. symmetric subsets.
- Shrinking some (Z^σ_s) by removing a family of disks. (hard!)
- Reducing T to $T - \tau$ if (Z^σ_s) stay away from almost singular parts.